Two pairs of symmetric optimal control problems in Economics *

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Abstract. Two optimal control problems arising in Economics are analyzed in order to elicit some information about other problems, symmetric with them. Different problems, based on the same economic model, can have very similar sets of necessary conditions. Such a situation occurs when considering different objectives for the same economic system. We prove that such conditions can be reduced to a controllability problem and that they concern the characterization of the Pareto efficient solutions of a multiobjective problem.

Keywords: Optimal control, Multiobjective programming, Pareto optimality

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1 Symmetries between problems

A lot of models arising in Economics are studied using optimal control theory. In this paper we want to analyze, from a different point of view, two classical problems: Ramsey’s model [5] and consumption versus investment model [6, p.78]. In both these models total utility is to be maximized, under the condition that a fixed final level for a state variable (e.g. the final value of the investment) is reached at least. Otherwise, it is interesting, from an economic point of view, to study also the (symmetric) problem, in which the state variable is to be maximized under the condition that the total utility reaches at least a fixed final level. In fact, both problems are quite similar

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and we can prove that the necessary conditions can be reduced to the same controllability problem. We might also formulate a third, two-objective problem, whose Pareto efficient solutions are characterized by the same necessary conditions. The result is essentially an application of the constraint method [2, pp.115–127] for generating noninferior solutions of a multiobjective problem.

The idea of analyzing symmetries among economical problems is well known in Consumer Theory, where the Marshall and Hicks problems are studied in a symmetric way using the nonlinear programming theory. The solutions one can obtain give a deeper understanding of both problems and the link between results permits an interesting economical analysis [7, 111-167], [1]. Some general results on the connections between the solutions of a non-linear programming problem and its reciprocal are given in [8]. One can find there, in particular, the discussion of sufficient conditions to assure that a primal problem and its reciprocal have the same solution, in a finite dimensional setting. Here we aim at proving similar results as those in [8] for two couples of symmetric optimal control problems, concerning two classical economic models, and at reading symmetry results in terms of a special Pareto efficient frontier. To the best of our knowledge, such symmetric approach is not frequently encountered in the economical models studied with the optimal control theory. In the Calculus of Variations the presence of symmetries between different problems is a well known subject [3, p.143] and, recently, such properties have been used to study a marketing problem [4]. This paper is meant to follow the same direction and show that one can reach a deeper knowledge of an economic model by considering symmetries among different problems.

The paper is organized as follows: in Section 2 we introduce the consumption versus investment model, we define the symmetric problem and we show that a controllability problem is the core of the necessary conditions for both problems. In Section 3 the same analysis is repeated for the Ramsey’s model.

2 Consumption versus investment

A country receives a continuous cash flow \( f(t) > 0 \) as aid during the programming interval \([0, T]\). Let \( U \) be a utility function
\( U(0) = 0, \ U'(0) = +\infty, \ U' > 0, \ \lim_{x \to +\infty} U'(x) = 0, \ U'' < 0, \ U \in C^2\) and let us denote by \( u(t) \in [0, 1] \) the part of the aid which is allocated to consumption; the decision represented by the control function \( u(t) \) produces a utility flow \( U(f(t)u(t)) \). If we define a state variable \( x_2(t) \) that represents the total utility obtained during the interval \([0, t] \), then its evolution is described by the equation \( \dot{x}_2(t) = U(f(t)u(t)) \). Let \( x_1(t) \) be the level of infrastructure at the time \( t \) in the country and let \((1 - u(t))\) be the part of the aid which is allocated to investment. The investment produces an increment in the level of infrastructure of the country, which is described by \( \dot{x}_1(t) = f(t)(1 - u(t)) \). The standard approach to this problem is to require that the government maximizes the total utility and reaches at least a fixed level of infrastructure at the end of the programming interval. On the other hand, also a different economical viewpoint may be interesting: the government might decide to maximize the final level of the infrastructure under the condition of reaching a fixed utility value. We want to study these problems in parallel and prove that the relevant necessary conditions lead to a controllability problem.

Let \( \alpha_T \) be the required minimum final level of the infrastructure and \( \alpha_0 \) be the initial level for the same state variable. The standard consumption versus investment problem can be formulated as follows (we will refer to this problem as the consumption problem):

\[
\max_{u \in L^1([0,T];[0,1])} x_2(T),
\]

s.t.

\[
\begin{aligned}
\dot{x}_1(t) &= f(t)(1 - u(t)) , \\
\dot{x}_2(t) &= U(f(t)u(t)) , \\
x_1(0) &= \alpha_0 , \\
x_2(0) &= 0 , \\
x_1(T) &\geq \alpha_T , \\
x_2(T) &\in \mathbb{R} .
\end{aligned}
\]  

(1)

Assuming the symmetric point of view, we consider that the country wants to reach a minimal level \( \beta_T \) of utility (fixed \textit{a priori} by the government) and it wants to maximize the final level of the infrastructure, starting from the initial level \( \alpha_0 \). Therefore, the investment
The problem can be defined as follows:

\[
\max_{u \in L^1([0,T];[0,1])} x_1(T),
\]

s.t.

\[
\begin{aligned}
\dot{x}_1(t) &= f(t)(1 - u(t)), \\
\dot{x}_2(t) &= U(f(t)u(t)), \\
x_1(0) &= \alpha_0, \\
x_1(T) &\in \mathbb{R}, \\
x_2(0) = 0, \\
x_2(T) &\geq \beta_T.
\end{aligned}
\] (2)

2.1 Necessary conditions

The Hamiltonian function for both problems (1) and (2) is the same:

\[
H(x_1, x_2, p_1, p_2, u, t) = p_1 f(t)(1 - u) + p_2 U(f(t)u).
\]

First of all, we notice that

\[
\begin{aligned}
H_u(x_1, x_2, p_1, p_2, u, t) &= -p_1 f(t) + p_2 f(t) U'(f(t)u), \\
H_{uu}(x_1, x_2, p_1, p_2, u, t) &= p_2 f^2(t) U''(f(t)u),
\end{aligned}
\]

hence the sign of the adjoint variable \( p_2 \) is connected with the concavity/convexity of the Hamiltonian function. The adjoint equations for both problems are the following:

\[
\begin{aligned}
\dot{p}_1(t) &= 0, \\
\dot{p}_2(t) &= 0.
\end{aligned}
\]

The transversality conditions are different and can be written for the consumption and the investment problem respectively as:

\[
\begin{aligned}
\begin{cases}
p_1(T) = \gamma_1, \\
p_2(T) = p_0 + \gamma_2, \\
\gamma_1 \geq 0, \\
\gamma_1 (x_1(T) - \alpha_T) = 0, \\
\gamma_2 = 0,
\end{cases} &\quad \begin{cases}
p_1(T) = p_0 + \gamma_1, \\
p_2(T) = \gamma_2, \\
\gamma_1 = 0, \\
\gamma_2 \geq 0, \\
\gamma_2 (x_2(T) - \beta_T) = 0.
\end{cases}
\end{aligned}
\]

2.2 Limit instances \((p_0 = 0)\)

In this subsection we look for a solution with \( p_0 = 0 \), as the necessary conditions require that \((p_0, \gamma_1, \gamma_2) \neq 0\) and \( p_0 \in \{0, 1\} \).
Consumption problem (1)
If \( p_0 = 0 \) and \( \gamma_2 = 0 \) then \( \gamma_1 > 0 \) must hold and therefore, by the transversality conditions, we have that \( x_1(T) = \alpha_T \). From the adjoint equations we obtain that \( p_1(t) \equiv \gamma_1 > 0 \) and \( p_2(t) \equiv 0 \), hence the Hamiltonian function becomes
\[
H(x_1, x_2, p_1, p_2, u, t) = \gamma_1 f(t) (1 - u),
\]
and is maximized w.r.t. the variable \( u \in [0, 1] \) if and only if \( u = 0 \). There exists a unique aid flow allocation that satisfies the maximum Hamiltonian condition, that of devoting all the aid to investment: such a solution is optimal if and only if \( \alpha_T = \alpha_{\text{max}} \), where
\[
\alpha_{\text{max}} = \alpha_0 + \int_0^T f(t) \, dt
\]
is the maximum reachable infrastructure value.

Investment problem (2)
If \( p_0 = 0 \) and \( \gamma_1 = 0 \) then \( \gamma_2 > 0 \) and therefore, by the transversality conditions, we get that \( x_2(T) = \beta_T \). The solutions of the adjoint equations are \( p_2(t) \equiv \gamma_2 > 0 \) and \( p_1(t) \equiv 0 \), hence the Hamiltonian function
\[
H(x_1, x_2, p_1, p_2, u, t) = \gamma_2 U(f(t) \, u)
\]
is maximized w.r.t. the variable \( u \in [0, 1] \) if and only if \( u = 1 \). Symmetrically to the first problem, we find a unique aid flow allocation and it consists in spending all the aid in consumption: it is optimal if and only if \( \beta_T = \beta_{\text{max}} \), where
\[
\beta_{\text{max}} = \int_0^T U(f(t)) \, dt
\]
is the maximum reachable immediate utility value.

2.3 Degenerate instances
If \( p_0 = 1 \), then the condition \( (p_0, \gamma_1, \gamma_2) \neq 0 \) is satisfied and the transversality conditions read, for problem (1) and problem (2) re-
respectively, as
\[
\begin{align*}
    p_1 (T) &= \gamma_1, \\
    p_2 (T) &= 1, \\
    \gamma_1 &\geq 0, \\
    \gamma_1 (x_1 (T) - \alpha_T) &= 0,
\end{align*}
\]
\[
\begin{align*}
    p_1 (T) &= 1, \\
    p_2 (T) &= \gamma_2, \\
    \gamma_2 &\geq 0, \\
    \gamma_2 (x_2 (T) - \beta_T) &= 0.
\end{align*}
\]

We consider first some particular choices of the parameters that turn the problems into trivial ones.

**Consumption problem** (1)
If \( \gamma_1 = 0 \), then from the adjoint equations we get \( p_1 (t) \equiv 0 \) and \( p_2 (t) \equiv 1 \), therefore the Hamiltonian function is
\[
H (x_1, x_2, p_1, p_2, u, t) = U (f (t) u),
\]
and is maximized w.r.t. \( u \in [0, 1] \) by choosing \( u = 1 \). The state function associated with the control \( u(t) \equiv 1 \) is admissible if and only if \( \alpha_T \leq \alpha_0 \). If all the aid is spent in consumption, then the infrastructure level remains constant and the final constraint \( x_1 (T) \geq \alpha_T \) is only satisfied when \( \alpha_0 \geq \alpha_T \).

**Investment problem** (2)
Symmetrically, if \( \gamma_2 = 0 \), then we obtain \( p_1 (t) \equiv 1 \) and \( p_2 (t) \equiv 0 \); therefore maximizing the Hamiltonian
\[
H (x_1, x_2, p_1, p_2, u, t) = f (t) (1 - u),
\]
w.r.t. \( u \in [0, 1] \), leads to the control \( u \equiv 0 \). The associated state function is admissible if and only if \( \beta_T \leq 0 \). If all the aid is invested, then the immediate utility does not increase and the final constraint \( x_2 (T) \geq \beta_T \) is not satisfied, unless \( 0 \geq \beta_T \), because \( U(0) = 0 \) and \( U' > 0 \).

### 2.4 The controllability problem
After excluding the previous instances, in which there was no trade off between consumption and investment, we obtain the following form for the transversality conditions of the problems (1) and (2):
\[
\begin{align*}
    \{ &p_1 (T) = \gamma_1 > 0, \\
    p_2 (T) = 1, \\
    x_1 (T) = \alpha_T, \}
\end{align*}
\]
\[
\begin{align*}
    \{ &p_1 (T) = 1, \\
    p_2 (T) = \gamma_2 > 0, \\
    x_2 (T) = \beta_T. \}
\end{align*}
\]
The adjoint functions $p_1(t)$ and $p_2(t)$ are strictly positive constants and such is also their ratio $\pi = p_1(t)/p_2(t) \in (0, +\infty)$. Hence the Hamiltonian function is strictly concave in $u$ and it is maximized in $\mathbb{R}$ at one point such that the first derivative w.r.t. $u$ vanishes:

$$U'(f(t)u) = p_1(t)/p_2(t) = \pi.$$ 

Now $U'$ has an inverse function $\phi : (0, +\infty) \to (0, +\infty)$ so that the above maximum condition is equivalent to $u = \phi(\pi)/f(t)$. Hence we obtain the control function

$$u^\pi(t) = \min \left\{ 1, \frac{\phi(\pi)}{f(t)} \right\}.$$

Now both problems are reduced to the same controllability problem. The government has to fix the constant $\pi \in (0, +\infty)$ in order to drive the system of ODEs

$$\begin{cases} 
\dot{x}_1(t) = f(t)(1 - u^\pi(t)), \\
\dot{x}_2(t) = U(f(t)u^\pi(t)),
\end{cases}$$

from the starting point

$$\begin{cases} 
x_1(0) = \alpha_0, \\
x_2(0) = 0,
\end{cases}$$

to a point in either final surface $S_1 = \{\alpha_T\} \times \mathbb{R}$, for the consumption problem (1), or $S_2 = \mathbb{R} \times \{\beta_T\}$, for the investment problem (2). Now, let us define

$$\bar{\pi} = \phi^{-1}(\max f) = U'(\max f),$$

which is the minimum observable marginal utility, then $\bar{\pi} > 0$ and

- if $\pi \leq \bar{\pi}$, we have the $u^\pi(t) \equiv 1$ and its associated solution is such that $x_1(T) = \alpha_0$, $x_2(T) = \beta_{\max}$;
- if $\pi \to +\infty$, the control $u^\pi(t)$ tends to $u^\infty(t) \equiv 0$ and $x_1(T) \to \alpha_{\max}$, $x_2(T) \to 0$. 


Moreover, the map
\[ \Xi : [\bar{\pi}, +\infty) \rightarrow [\alpha_0, \alpha_{\text{max}}] \times (0, \beta_{\text{max}}] \]
\[ \pi \mapsto \left( \alpha_0 + \int_0^T [f(t) - \phi(\pi)]^+ \, dt, \int_0^T U(\min\{f(t), \phi(\pi)\}) \, dt \right) \]
is continuous, its first component \( \xi_1(\pi) \) is strictly increasing, whereas its second component \( \xi_2(\pi) \) is strictly decreasing. Hence \( \Xi \) is a 1-1 function. Given a point \( \Xi(\pi) = (\xi_1(\pi), \xi_2(\pi)) \) on its image, we can state that

- \( \xi_1(\pi) \) is the optimal value of \( x_1(T) \) in the investment problem (2) with \( \beta_T = \xi_2(\pi) \);
- \( \xi_2(\pi) \) is the optimal value of \( x_2(T) \) in the consumption problem (1) with \( \alpha_T = \xi_1(\pi) \).

### 2.5 Pareto efficient solutions

In fact we might think of a third situation, in which the country government has not fixed \textit{a priori} any infrastructure nor any utility value threshold, but aims at maximizing the two-dimensional objective \( (x_1(T), x_2(T)) \). From this viewpoint, the country government is interested in the Pareto efficient solutions with respect to the criteria infrastructure value, represented by \( x_2(T) \) and immediate utility, represented by \( x_1(T) \). Now, solving the controllability problem is equivalent to determining the Pareto efficient solutions.

Let us define the functions
\[ V_2 : [\alpha_0, \alpha_{\text{max}}] \rightarrow \mathbb{R}, \quad V_1 : [0, \beta_{\text{max}}] \rightarrow \mathbb{R}, \]
\[ \alpha_T \mapsto V_2(\alpha_T) = x_2^*(T), \quad \beta_T \mapsto V_1(\beta_T) = x_1^*(T), \]
as the optimal values of the objective functionals of the consumption problem, \( V_2(\alpha_T) \), which we call the \textit{immediate utility of infrastructure value}, and of the investment problem, \( V_1(\beta_T) \), which we call the \textit{infrastructure value of immediate utility}.

We notice that \( V_2(\alpha_T) \) is the final value of the state function \( x_2(t) \), associated with the control \( u^\pi(t) \), where \( \pi = \pi_1(\alpha_T) \) satisfies the condition
\[ \int_0^T [f(t) - \phi(\pi)]^+ \, dt = \alpha_T - \alpha_0. \]
Hence $\pi_1(\alpha_T)$ is a continuous and strictly decreasing function of $\alpha_T$ and thus also $V_2(\alpha_T)$ is continuous and strictly decreasing.

On the other hand, $V_1(\beta_T)$ is the final value of the state function $x_1(t)$, associated with the control $u^\pi(t)$, where $\pi = \pi_2(\beta_T)$ satisfies the condition

$$\int_0^T U(\min\{f(t), \phi(\pi)\}) \, dt = \beta_T.$$ 

Hence $\pi_2(\beta_T)$ is a continuous and strictly decreasing function of $\beta_T$ and thus also $V_1(\beta_T)$ is continuous and strictly decreasing.

From the definition of the map $\Xi$ in the previous Section, we notice that

- if $\pi = \pi_1(\alpha_T)$, then
  $$\xi_2(\pi) = V_2(\alpha_T), \quad \xi_1(\pi) = \alpha_T;$$
- if $\pi = \pi_2(\beta_T)$, then
  $$\xi_1(\pi) = V_1(\beta_T), \quad \xi_2(\pi) = \beta_T.$$ 

Hence, for all $\pi \in [\bar{\pi}, +\infty)$, we have that

$$\xi_1(\pi) = V_1(\xi_2(\pi)), \quad \xi_2(\pi) = V_2(\xi_1(\pi)),$$

i.e. $V_1 = V_2^{-1}$. In other words, we can state the following identity results for the optimal controls of both problems 1 and 2:

- if $u^*(t)$ is an optimal control of the consumption problem, then it is optimal also for the investment problem, provided that $\beta_T = V_2(\alpha_T)$;
- if $u^*(t)$ is an optimal control of the investment problem, then it is optimal also for the consumption problem, provided that $\alpha_T = V_1(\beta_T)$.

3 Ramsey’s model

The economic growth theory, which is based on the pioneering work of Ramsey [5], is a cornerstone of economic modeling. Let us define:
$x_1(t), x_2(t)$ the state functions, i.e. the capital value at time $t$ and the total utility produced during the interval $[0, t]$;
$u(t)$ the control function, i.e. the consumption flow at time $t$;
$f(x_1)$ the production function, strictly positive, strictly increasing and concave, that represents the production flow associated with the value $x_1$ of the capital stock;
$U(c)$ the utility function, strictly increasing and strictly concave, that describes the utility flow associated with the rate $c$ of consumption ($U(0) = 0, U'(0) = +\infty, U' > 0, \lim_{c \to +\infty} U'(c) = 0, U'' < 0, U \in C^2$);
$\rho \geq 0$ the discount factor.

Therefore the standard Ramsey’s problem is formulated as follows:

$$
\max_{u \in L^1([0,T];[0,\infty))} x_2(T), \quad \text{s.t.} \quad
\begin{aligned}
\dot{x}_1(t) &= f(x_1(t)) - u(t), \\
\dot{x}_2(t) &= e^{-\rho t} U(u(t)), \\
x_1(0) &= \alpha_0, \\
x_2(0) &= 0,
\end{aligned}
$$

(3)

The decision maker controls the consumption and wants to maximize the final total utility, subject to a lower bound $\alpha_T$ on the capital stock level at the end of the programming interval ($\alpha_0$ is the capital level at time $t = 0$).

Assuming the symmetric point of view, the decision maker has to reach a minimal utility level $\beta_T$ (a priori fixed) and wants to maximize the final capital stock level. Therefore, the symmetric Ramsey’s problem is

$$
\max_{u \in L^1([0,T];[0,\infty))} x_1(T), \quad \text{s.t.} \quad
\begin{aligned}
\dot{x}_1(t) &= f(x_1(t)) - u(t), \\
\dot{x}_2(t) &= e^{-\rho t} U(u(t)), \\
x_1(0) &= \alpha_0, \\
x_2(0) &= 0,
\end{aligned}
$$

(4)

Our aim is to reduce both problems to a suitable controllability problem, as we have done in the case of the consumptions versus investment model.
3.1 Necessary conditions

The Hamiltonian function for both problems is
\[ H(x_1, x_2, p_1, p_2, u, t) = p_1(f(x_1) - u) + p_2e^{-\rho t}U(u) \]
and its first and second derivatives w.r.t. \( u \) are
\[ H_u = -p_1 + p_2e^{-\rho t}U'(u), \quad H_{uu} = p_2e^{-\rho t}U''(u). \]

We obtain the adjoint equations
\[
\begin{align*}
\dot{p}_1(t) &= -p_1(t)f'(x_1(t)), \\
\dot{p}_2(t) &= 0,
\end{align*}
\]
and the transversality conditions for the problems (3) and (4) respectively
\[
\begin{align*}
\left\{ \begin{array}{l}
p_1(T) = \gamma_1, \\
p_2(T) = p_0 + \gamma_2, \\
\gamma_1 \geq 0, \\
\gamma_1(x_1(T) - \alpha_T) = 0, \\
\gamma_2 = 0,
\end{array} \right. & \quad \begin{array}{l}
p_1(T) = p_0 + \gamma_1, \\
p_2(T) = \gamma_2, \\
\gamma_1 = 0, \\
\gamma_2 \geq 0, \\
\gamma_2(x_2(T) - \beta_T) = 0.
\end{array}
\end{align*}
\]

Without any further assumption on the production function we cannot continue the analysis, because the motion equation and the first adjoint equation are coupled. Therefore we focus on a special case, by adding the assumption that the production function is linear:
\[ f(x_1) = \lambda x_1, \quad \text{where } \lambda > 0. \]

As a consequence, the first adjoint equation is \( \dot{p}_1(t) = -\lambda p_1(t) \) and the adjoint system is uncoupled from the motion equation, so that the system results analytically manageable.

3.2 Limit instances \((p_0 = 0)\)

We first consider the possibility that \( p_0 = 0 \).

Problem (3)
If \( p_0 = 0 \) and \( \gamma_2 = 0 \) then \( \gamma_1 > 0 \) and therefore the transversality condition implies that \( x_1(T) = \alpha_T \). The solutions of the adjoint
equations are \( p_1(t) = \gamma_1 \exp(\lambda(T-t)) > 0 \) and \( p_2(t) \equiv 0 \), hence the Hamiltonian function

\[
H(x_1, x_2, p_1, p_2, u, t) = p_1(\lambda x_1 - u)
\]
is maximized w.r.t. \( u \in [0, +\infty) \) by choosing \( u = 0 \). The resulting no consumption policy satisfies the transversality condition if and only if the associated capital stock level reaches the final value \( x_1(T) = \alpha_T \). This is true if and only if \( \alpha_T = \alpha_0 e^{\lambda T} \).

**Problem (4)**

Symmetrically, if \( p_0 = 0 \) and \( \gamma_1 = 0 \) then \( \gamma_2 > 0 \), hence we get \( x_2(T) = \beta_T \). By the adjoint equations we have that \( p_2(t) \equiv \gamma_2 > 0 \) and \( p_1(t) \equiv 0 \), hence

\[
H(x_1, x_2, p_1, p_2, u, t) = p_2 e^{-\rho t} U(u)
\]
which is upper unbounded w.r.t. \( u \in [0, +\infty) \). As the shadow price of the capital stock \( p_1(t) \) is zero, unbounded consumption is suggested: there does not exist any optimal solution with \( p_0 = 0 \).

### 3.3 Degenerate instances

Now, let us assume that \( p_0 = 1 \); the transversality conditions for the problems (3) and (4) become the following, respectively:

\[
\begin{align*}
\begin{cases}
p_1(T) = \gamma_1, \\
p_2(T) = 1, \\
\gamma_1 \geq 0, \\
\gamma_1 (x_1(T) - \alpha_T) = 0,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
p_1(T) = 1, \\
p_2(T) = \gamma_2, \\
\gamma_2 \geq 0, \\
\gamma_2 (x_2(T) - \beta_T) = 0.
\end{cases}
\end{align*}
\]

We consider first the trivial cases.

**Problem (3)**

If \( \gamma_1 = 0 \), then the adjoint equations give us \( p_1(t) \equiv 0 \) and \( p_2(t) \equiv 1 \); the Hamiltonian function becomes

\[
H(x_1, x_2, p_1, p_2, u, t) = e^{-\rho t} U(u),
\]
which is upper unbounded w.r.t. \( u \). This is the same situation as with \( p_0 = 0 \) for problem (4): there does not exist any optimal solution with \( p_0 = 1 \) and \( \gamma_1 = 0 \).
Problem (4)
Symmetrically we have: if $\gamma_2 = 0$, then the solution of the adjoint equations are $p_1 (t) = \exp (\lambda (T - t)) > 0$ and $p_2 (t) \equiv 0$; hence
\[ H (x_1, x_2, p_1, p_2, u, t) = p_1 (\lambda x_1 - u) , \]
which is maximized w.r.t. $u \in [0, +\infty)$ by choosing $u = 0$. The resulting control $u(t) \equiv 0$ satisfies the final conditions if and only if $\beta_T \leq 0$, in which case only capital growth matters.

3.4 The controllability problem
Focusing on the nontrivial cases only, we arrive at the transversality conditions that can be written for the problems (3) and (4) respectively as
\[
\begin{cases}
p_1 (T) = \gamma_1 > 0 , \\
p_2 (T) = 1 , \\
x_1 (T) = \alpha T ,
\end{cases}
\quad \begin{cases}
p_1 (T) = 1 , \\
p_2 (T) = \gamma_2 > 0 , \\
x_2 (T) = \beta T .
\end{cases}
\]
These conditions, together with the adjoint equations, imply that $p_1 (t)$ and $p_2 (t)$ are strictly positive functions and their analytical forms for problem (3) and (4) respectively are:
\[
\begin{cases}
p_1 (t) = \gamma_1 e^{\lambda (T-t)} > 0 , \\
p_2 (t) \equiv 1 ,
\end{cases}
\quad \begin{cases}
p_1 (t) = e^{\lambda (T-t)} > 0 , \\
p_2 (t) \equiv \gamma_2 > 0 .
\end{cases}
\]
Now, we observe that the Hamiltonian function, which has to be maximized w.r.t. $u \in [0, +\infty)$, is a strictly concave function. Hence it has a unique maximum point, which is the solution of the equation
\[ U'' (u) = e^{\rho t} p_1 (t) / p_2 (t) . \] (5)
After denoting by $\phi : (0, +\infty) \to (0, +\infty)$ the inverse function of $U''$, we obtain that the condition (5) implies that the optimal policy is
\[ u^* (t) = \phi (\pi e^{\rho t - \lambda t}) , \]
where $\pi$ is a positive constant. Actually, the decision maker has to choose the constant $\pi \in (0, +\infty)$ in order to drive the system of ODEs
\[
\begin{cases}
\dot{x}_1 (t) = \lambda x_1 (t) - u^* (t) , \\
\dot{x}_2 (t) = e^{-\rho t} U (u^* (t)) ,
\end{cases}
\]
from the initial point
\[
\begin{align*}
    x_1(0) &= \alpha_0, \\
    x_2(0) &= 0,
\end{align*}
\]
to a point in either final surface \( S_1 = \{\alpha_T\} \times \mathbb{R} \), for the standard problem (3), or \( S_2 = \mathbb{R} \times \{\beta_T\} \), for the symmetric problem (4).

### 3.5 Pareto efficient solutions

Following the logical path of Section 2.5, let us define the functions \( V_2(\alpha_T) \) and \( V_1(\beta_T) \) as the optimal values of the objective functionals of the standard Ramsey’s problem, the former, which we call the immediate utility of final capital stock, and of the symmetric Ramsey’s problem, the latter, which we call the final capital stock level of immediate utility.

We note that the optimal control in both problems is characterized only by the value of the parameter \( \pi \). Moreover, if we require that the system reaches an admissible final value for the state variable \( x_1^*(T) \) (\( x_2^*(T) \), respectively), then we find a unique \( \pi \) which allows to satisfy the requirement. Such an observation allows to consider \( x_1^*(T) \) and \( x_2^*(T) \) as two 1-1 functions of \( \pi \) hence the relation between \( x_1^*(T) \) and \( x_2^*(T) \) is a 1-1 function, which is \( V_1 \) in one direction and \( V_2 \) in the opposite one. Thus \( V_2 = V_1^{-1} \). Their regularity follows from the utility function regularity. Finally,

- if \( u^*(t) \) is an optimal control of the standard Ramsey’s problem, then it is optimal also for the symmetric Ramsey’s problem, provided that \( \beta_T = V_2(\alpha_T) \);
- if \( u^*(t) \) is an optimal control of the symmetric Ramsey’s problem, then it is optimal also for the standard Ramsey’s problem, provided that \( \alpha_T = V_1(\beta_T) \).

### 4 Conclusion

In this paper we consider some symmetric problems in the framework of two classical economic models (i.e. consumption versus investment and Ramsey’s models). We prove that, from an analytical point of view, both the standard problem and the symmetric one can be reduced to the same controllability problem. The exploitation of the
symmetry among problems is well known and studied in the mathematical optimization theory, but it is met less frequently in the economic analysis. Here we have shown that the solutions of two classical problems can be reinterpreted as the solutions of different problems which are symmetric to them and which have their own interesting economic meaning.

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References