On Boundary Control Problems in Slow Processes for Piezothermoelastic Plates

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Abstract

We consider a piezothermoelastic panel occupied by a material of hexagonal crystal class. We study the response when the boundary conditions vary very slowly with time and one of the bounding faces is subject to thermal exposure. We show that in some cases the temperature on the other bounding face can be controlled by the difference of electric potential between the faces.

Key words: Piezothermoelasticity, Electromagnetic effects, Thermal effects, Plates, Piezothermoelastic plate

1 Introduction

1.1 Premise

Piezoelectricity is the property of generating an electric field (mechanical stress) in response to an applied mechanical stress (electric field); pyroelectricity is the property of generating an electric field (temperature change) by a temperature change (electric field).

For a piezothermoelastic body a natural problem, useful for practical applications, is to study the electromechanical effects due to a prescribed temperature on part of the boundary.
The present paper is a starting study of boundary control problems in plate-like bodies exhibiting pyroelectricity. This in order to theoretically establish whether and under which boundary conditions a given boundary temperature and/or electric potential can be usefully employed to obtain some type of control, e.g. on temperature and/or electric potential, in a given material surface. This theoretic study may be useful with regard to real panels subject to sun exposure or lying in contact with an external heat source, in order e.g. to passively exploit this boundary condition.

Here we define first a general boundary control problem and then study the particular problem of a plate occupied by a material exhibiting piezoelectric and pyroelectric properties.

1.2 Static Boundary Value Problem for a Piezothermoelastic body

Here we adopt the summation convention and comma notation for partial derivatives, so that $a^i b_i = \sum a^i b_i$ and $f_{,i} = \partial f / \partial x_i$.

Consider a piezothermoelastic body $B$ that, in the reference configuration, occupies a region $V$ with boundary surface $S$. The deformation of the body is described by

$$y_i = y_i(X_A) = \delta^B_i X_B + u_i(X_A),$$

where $y_i$ denote the spatial coordinates and $X_L$ the reference coordinates of material points with respect to the same Cartesian coordinate system.

The Piola-Kirchhoff stress tensor, electric displacement vector, and heat flux vector are respectively given by constitutive functions

$$t_{Li} = \hat{t}_{Li}(\theta, E_i, E_{AB}, X_A), \quad \text{(1)}$$
$$D_L = \hat{D}_L(\theta, E_i, E_{AB}, X_A), \quad q_L = \hat{q}_L(\theta, E_i, E_{AB}, X_A), \quad \text{(2)}$$

where $\theta, \phi, E_i = -\phi_{,i}$, $E_{AB} = (y_{i,A}y_{i,B} - \delta_{AB})/2$ are the absolute temperature, electric potential, electric field and strain tensor, respectively.

Balance law of linear momentum, Maxwell’s equation, and balance law of conservation of energy, respectively lead to the equilibrium relations

$$t_{Li, L} + \rho_o f_l = 0, \quad D_{L, L} = \rho_e, \quad q_{L, L} = \rho_o \gamma, \quad \text{(3)}$$

where $\rho_o$ is the mass-density in the reference configuration, $f_l$ is the body force per unit mass, $\rho_e$ is the body free charge density, and $\gamma$ is the body heat source per unit mass.
To describe the corresponding boundary conditions, three partitions \((S_{i1}, S_{i2})\), \(i = 1, 2, 3\), of the boundary surface \(S = \partial B\) can be assigned. For mechanical boundary conditions, displacement \(\mathbf{u}\) and traction \(\mathbf{t}\) per unit undeformed area are prescribed, respectively, on \(S_{11}\) and \(S_{12}\); for electric boundary conditions, electric potential \(\phi\) and surface-free charge \(D\) per unit undeformed area are prescribed, respectively, on \(S_{21}\) and \(S_{22}\); while for thermic boundary conditions, temperature \(T\) and normal heat flux \(\mathbf{q}\) per unit undeformed area are prescribed, respectively, on \(S_{31}\) and \(S_{32}\). Hence, we can write

\[
\begin{align*}
  u_i &= \mathbf{u}_i \text{ on } S_{11}, \quad t_{Li}N_L = \mathbf{t}_i \text{ on } S_{12} \quad ('\text{mechanical}') , \\
  \phi &= \phi \text{ on } S_{21}, \quad D_LN_L = D \text{ on } S_{22} \quad ('\text{electric}') , \\
  T &= T \text{ on } S_{31}, \quad q_LN_L = q \text{ on } S_{32} \quad ('\text{thermic}') , \\
  S_{11} \cup S_{12} &= S , \quad S_{11} \cap S_{12} = \emptyset \quad (i = 1, 2, 3),
\end{align*}
\]

where \(N = (N_L)\) is the unit exterior normal on \(S\) and \(T = \theta - \theta_o\) is the incremental temperature with respect to the temperature \(\theta_o\) in the reference state.

The boundary value problem is then stated as: to find the solution \((\phi, T, \mathbf{u})\) in \(B\) to the constitutive relations (1), (2) and field equilibrium equations (3) which satisfies the boundary conditions (4)-(6) for given \(\mathbf{u}_i, \mathbf{t}_i, \phi, D, T, \mathbf{q}\).

Of course, existence and uniqueness of the solution must be separately examined, but here are assumed.

1.3 Boundary Control Problem

Let \(S_o\) be a regular surface contained in \(\overline{B} := B \cup \partial B\) (possibly \(S_o \subset \partial B\)), with oriented normal unit vector \(N\), let \(C : S_o \to \mathbb{R}\) be a scalar smooth function, and let \(\chi\) be any one of the ten quantities

\[
\phi, \quad \theta, \quad D_LN_L, \quad q_LN_L, \quad u_i, \quad t_{Li}N_L \quad (i = 1, 2, 3).
\]

The boundary control problem for \(\chi\) on \(S_o\) with goal \(C\), (abbreviated to \(BCP \chi|_{S_o} = C\)), is the following: For arbitrary \(n < 10\) boundary conditions of the type (4)-(6), choose the remaining \(10 - n\) boundary conditions in (4)-(6) such that the solution \((\phi, T, \mathbf{u})\) to the corresponding boundary value problem yields \(\chi|_{S_o} = C\).
1.4 The Boundary Control Problems solved here

In the present paper, we solve some boundary control BCP, for certain linear piezothermoelastic bodies, that occupy a plate $P$ infinite in extent and bounded by two parallel planes. The plate has a natural equilibrium state, i.e., with no initial field, and is occupied by a heat-conducting piezoelectric material with the symmetry of the hexagonal crystal class $C_{6v} = 6mm$, so that ferroelectric ceramics are included. We assume that $P$ is subject to a constant temperature on the upper face that in effect may vary slowly with time. On the lower face, the displacement is prescribed, as when, for example, $P$ is welded to a fixed flat body.

We study processes which are homogeneous on each plane parallel to the boundary planes, that is, they depend only on the thickness coordinate, and, moreover, vary very slowly with time. The precise equilibrium boundary value problems studied are summarized in the table and are completely solved once their exact solutions are determined.

In these problems, we take $n = 9$ and $S_o$ any fixed plane parallel to the plane boundaries of the plate. In BCP.s I.1.3 and I.3.3 either $\chi = T$ or $\chi = \phi$, i.e.,
either temperature or electric potential can be controlled on \( S_0 \).

In BCP's II.1.3 and II.3.3 we have \( \chi = T \), i.e. whatever temperature is prescribed at the upper face, the temperature can be controlled on \( S_0 \) by the electric potential difference between the two bounding planes.

2 Linear Piezothermoelasticity

2.1 Linear constitutive equations

The linear constitutive equations are specified below in terms of the constitutive coefficients: \( c_{klij} = \) elastic moduli; \( e_{ijkl} = \) piezoelectric moduli; \( \beta_{kl} = \) thermal stress moduli; \( \kappa_E^E = \) dielectric susceptibility; \( q_k = \) pyroelectric polarizability; \( \varepsilon_{kl} = \) permittivity moduli; \( \kappa_{kl} = \) Fourier coefficients; \( \gamma = \) heat capacity; \( \eta_o = \) entropy at the natural state; \( \rho_o = \) mass-density at the natural state. These coefficients, each assumed to be constant, satisfy the following symmetry conditions:

\[
\begin{align*}
\text{symmetry conditions} & : c_{klij} = c_{ijkl} = c_{lkij} = c_{klji} , \\
\beta_{ij} = \beta_{ji} , & \quad \kappa_{kl} = \kappa_{lk} , & \quad \kappa_E^E_{kl} = \kappa_E^E_{lk} .
\end{align*}
\]

With respect to a natural reference state, i.e., a state free from mechanical and/or electric fields, and with constant temperature \( \theta_o \), we assume the following standard constitutive equations [1]-[5, p.122], (6)), respectively for the stress tensor, electric displacement vector and heat flux vector:

\[
\begin{align*}
t_{kl} &= c_{klij} u_{i,j} - e_{ijkl} E_i - \beta_{kl} T , \\
D_k &= e_{kij} u_{i,j} + \varepsilon_{kl} E_i + \tilde{\omega}_k T , & \quad q_k = - \kappa_{kl} T_{,l} - \kappa_E^E_{kl} E_l ,
\end{align*}
\]

where \( E_i = -\phi_{,i} , \quad T = \theta - \theta_o \) is the incremental absolute temperature with respect to the absolute temperature \( \theta_o \) in the natural reference state.

2.2 Field equations of equilibrium

The linearized field equations of equilibrium, obtained by substituting the constitutive equations (10)-(11) in the balance laws (3) taking \( f_k = \rho_e = \gamma = 0 \) are given by

\[
\begin{align*}
c_{klij} u_{i,j,k} + e_{ijl} \phi_{,ij} - \beta_{kl} T_{,k} = 0 & \quad (l = 1, 2, 3) ,
\end{align*}
\]
\[ e_{kji} u_{j,ik} - \varepsilon_{kj} \phi_{jk} + \dot{\omega}_k T_{i,k} = 0, \quad -\kappa_{kj} T_{j,ik} + \kappa_{jk}^E \phi_{jk} = 0. \quad (13) \]

### 2.3 Use of compressed notation and matrix arrays

As is well known, the matrix notation consists of replacing \( ij \) or \( kl \) by \( p \) or \( q \), where \( i, j, k, l \) take the values 1, 2, 3 and \( p, q \) take the values 1, 2, 3, 4, 5, 6 according to the following relations:

<table>
<thead>
<tr>
<th>( ij ) or ( kl )</th>
<th>11</th>
<th>22</th>
<th>33</th>
<th>23 or 32</th>
<th>31 or 13</th>
<th>12 or 21</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p, q )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

By virtue of the above identification, the constitutive equations become

\[ t_p = c_{pq} S_q - e_{ip} E_i - \beta_p T, \quad (14) \]
\[ D_i = e_{iq} S_q + \varepsilon_{ik} E_i + \dot{\omega}_i T, \quad q_i = -\kappa_{il} T_{i,l} - \kappa_{il}^E E_l, \quad (15) \]

where \( S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \), \( S_{ij} = S_p \) when \( i = j = p = 1, 2, 3 \), \( 2S_{ij} = S_p \) when \( i \neq j \), \( p = 4, 5, 6 \), and \( e_{ikl} = e_{ip} \).

### 3 Hexagonal materials

#### 3.1 Constitutive equations for ferroelectric ceramics

The polarized ferroelectric ceramics have the symmetry of a hexagonal crystal in class \( C_{6v} = 6mm \). Choosing \( x_3 \) in the polarization direction, assuming \( \beta_{ij} = 0 \) for \( i \neq j \) and putting

\[ \beta_k := \beta_{kk} \quad (k = 1, 2, 3), \quad [\beta_p] = \begin{bmatrix} \beta_1, \beta_2, \beta_3, 0, 0, 0 \end{bmatrix}, \quad (16) \]

the constitutive equations (10) and (11) become (cf. e.g. [6], [7, p.58], and [1])

\[ t_1 = c_{11} u_{1,1} + c_{12} u_{2,2} + c_{13} u_{3,3} + e_{31} \phi_{3,3} - \beta_1 T, \]
\[ t_2 = c_{12} u_{1,1} + c_{11} u_{2,2} + c_{13} u_{3,3} + e_{31} \phi_{3,3} - \beta_2 T, \]
\[ t_3 = c_{13} (u_{1,1} + u_{2,2}) + c_{33} u_{3,3} + e_{33} \phi_{3,3} - \beta_3 T, \]
\[ t_4 = c_{44} (u_{3,2} + u_{2,3}) + e_{15} \phi_{2,2}, \quad (17) \]
3.2 Field equations of equilibrium

When the constitutive relations (17), (18) are substituted in the equilibrium field equations (12)-(13) we have

\[ D_1 = e_{15} (u_{3,1} + u_{1,3}) - \varepsilon_{11} \phi_{,1} + \tilde{\omega}_1 T, \]
\[ D_2 = e_{15} (u_{3,2} + u_{2,3}) - \varepsilon_{11} \phi_{,2} + \tilde{\omega}_2 T, \]
\[ D_3 = e_{31} (u_{1,1} + u_{2,2}) + e_{33} u_{3,3} - \varepsilon_{33} \phi_{,3} + \tilde{\omega}_3 T. \]  

(18)

Particular forms of the solution to these equations are discussed in the next section, while general expressions are derived in the Appendix.

4 Quasi-statics

A principal application of the present theory is to a pyroelectric plate bonded to a fixed foundation, with the upper plane face exposed to sunlight. For this
the boundary conditions include (i) the prescription of temperature on the
upper bounding plane, and (ii) the condition of assigned displacement on the
lower bounding plane. Furthermore, the prescribed boundary values may be
understood to be functions of a parameter $\tau$ which depends slowly on time:

$$\tau = \tau(t), \quad |\tau'(t)| \text{ small}.$$

Hence, we refer to equations (19)-(23) augmented by these slowly varying
boundary conditions as a boundary value problem of quasi-statics.

### 4.1 Boundary Control Problem I.1.3

#### 4.1.1 Statement of the problem

The plate $P$ is bounded by the parallel planes $x_1 = \pm h$ and is coated by an
infinitesimally thin electrode on the plane $x_1 = h$, so that all its mechanical
effects may be ignored. We seek solutions of the form

$$T = T(x_1), \quad \phi = \phi(x_1), \quad u_i = u_i(x_1),$$

which when substituted in (19)-(23) give

$$-\beta_1 T_{,1} + c_{11} u_{1,11} = 0, \quad c_{66} u_{2,11} = 0,$$

$$c_{44} u_{3,11} + e_{15} \phi_{,11} = 0,$$

$$e_{15} u_{3,11} + \tilde{\omega}_1 T_{,1} - \varepsilon_{11} \phi_{,11} = 0, \quad -\kappa_{11} T_{,11} + \kappa_{11}^{E} \phi_{,11} = 0.$$

**BCP I.1.3** : To find the solution of the form (24) to the field equations (25)-
(27), subject to the ten boundary conditions

$$T(h) = \bar{T}, \quad \phi(h) = \bar{\phi}, \quad t_1(h) = \bar{t}_1, \quad t_6(h) = \bar{t}_2, \quad t_5(h) = \bar{t}_3,$$

$$u_i(-h) = \bar{u}_i \quad (i = 1, 2, 3), \quad -D_1(-h) = \bar{D}, \quad -q_1(-h) = \bar{q},$$

with $\bar{T}, \bar{\phi}, \bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{D}$ and $\bar{q}$ assigned real constants.

#### 4.1.2 General solution of BCP I.1.3

By Proposition 6.1, with

$$c = c_{44}, \quad e = e', \quad \omega = \tilde{\omega}_1, \quad \varepsilon = \varepsilon_{11}, \quad \kappa = \kappa_{11}, \quad \kappa' = \kappa_{11}^{E}, \quad \beta = 0,$$
\( K = k/k', \ A = c\omega, \ B = \epsilon e' + c\epsilon, \ a = AK^{-1}B^{-1}, \) \hspace{1cm} (30)

we have from (100) that the general solution to equations (26)-(27) is

\[ u_3(x_1) = -c^{-1} eKT_1 e^{ax_1} + U_{31} x_1 + U_{32}, \] \hspace{1cm} (31)

\[ T(x_1) = T_1 e^{ax_1} + T_2, \] \hspace{1cm} (32)

\[ \phi(x_1) = KT_1 e^{ax_1} + F_1 x_1 + F_2, \] \hspace{1cm} (33)

where \( T_1, T_2, F_1, F_2, U_{31}, U_{32} \) are arbitrarily chosen smooth functions of \( \tau. \)

Further, (32) implies \( T_1 = T_1 ae^{ax_1} \) which together with (25)_1 yields

\[ u_1(x_1) = a^{-1} c^{-1} \beta_1 T_1 e^{ax_1} + U_{11} x_1 + U_{12}, \] \hspace{1cm} (34)

while (25)_2 gives

\[ u_2(x_1) = U_{21} x_1 + U_{22}, \] \hspace{1cm} (35)

where \( U_{\alpha\beta}, \alpha, \beta = 1, 2, \) are arbitrary smooth functions of \( \tau. \)

4.1.3 Decomposition of BCP I.1.3

We solve BCP I.1.3 by decomposition into two parts, described below.

Part 1 of BCP I.1.3. We first consider the boundary conditions

\[ T(h) = T, \quad \phi(h) = \phi, \quad t_5(h) = \bar{t}_3, \]
\[ u_3(-h) = \bar{u}_3, \quad -D_1(-h) = \bar{D}, \quad -q_1(-h) = \bar{q}. \] \hspace{1cm} (36)

Note that by Eq.s (18), (31)-(33) and (101), the 5-th boundary condition above becomes

\[ -e u_{3,1} + \epsilon \phi_{,1} - \omega T = -\omega T_2 + \epsilon F_1 - e U_{31} = \bar{D}, \] \hspace{1cm} (37)

and by (15)_2 the 6-th boundary condition above respectively

\[ \kappa T_{,1} - \kappa' \phi_{,1} = -\kappa' F_1 = \bar{q}. \] \hspace{1cm} (38)

Now (31)-(33) satisfy (36) provided

\[ T_1 e^{ah} + T_2 = T, \quad KT_1 e^{ah} + F_1 + F_2 = \bar{\phi}, \] \hspace{1cm} (39)
\[ eF_1 + cU_{31} = \bar{r}_3, \quad -e^{-1}eKT_1 e^{-ah} - U_{31}h + U_{32} = \bar{u}_3, \quad (40) \]
\[ -\omega T_2 - eU_{31} + \varepsilon F_1 = \bar{D}, \quad -\kappa^l F_1 = \bar{q}. \quad (41) \]

By solving the above system of equations in the unknowns \((T_1, T_2, F_1, F_2, U_{31}, U_{32})\), we obtain
\[ T_2 = -\frac{c\bar{D} + e\bar{r}_3}{\varepsilon \omega} - K \frac{c\varepsilon + e^2}{\omega k} \bar{q}, \quad T_1 = e^{-ah}(T - T_2), \quad (42) \]
\[ F_1 = -Kk^{-1} \bar{q}, \quad F_2 = \bar{\phi} - hF_1 - Ke^ahT_1, \quad (43) \]
\[ U_{31} = \frac{1}{c}(\bar{r}_3 - eF_1), \quad U_{32} = \bar{u}_3 - hU_{31} + \frac{Ke}{c\varepsilon e^ah} T_1, \quad (44) \]

so that the solution to the first part of BCP I.1.3 becomes in particular
\[ \phi(x_1) = K\left(e^{a(x_1-h)} - 1\right)\left(T + \frac{c\bar{D} + e\bar{r}_3}{\varepsilon \omega} + K \frac{c\varepsilon + e^2}{\omega k} \bar{q}\right) + \bar{\phi} + \frac{2h}{k'} \bar{q}, \quad (45) \]
\[ T(x_1) = \left(e^{a(x_1-h)} - 1\right)\left(\frac{c\bar{D} + e\bar{r}_3}{\varepsilon \omega} + K \frac{c\varepsilon + e^2}{\omega k} \bar{q}\right) + e^{a(x_1-h)}T. \quad (46) \]

Hence,
\[ \phi(-h) = K\left(e^{-2ah} - 1\right)\left(T + \frac{c\bar{D} + e\bar{r}_3}{\varepsilon \omega} + K \frac{c\varepsilon + e^2}{\omega k} \bar{q}\right) + \bar{\phi} + \frac{2h}{k'} \bar{q}, \quad (47) \]
\[ T(-h) = \frac{e^{-2ah} - 1}{\varepsilon \omega} \left[c\bar{D} + e\bar{r}_3 + Kk^{-1}(c\varepsilon + e^2) \bar{q}\right] + e^{-2ah}\bar{T}, \quad (48) \]

and for \(0 = \bar{D} = \bar{r}_3 = \bar{q}\) we have
\[ \phi(-h) = \bar{\phi} + K\left(e^{-2ah} - 1\right)T, \quad T(-h) = e^{-2ah}\bar{T}, \quad (49) \]

which yield electric potential and temperature in the plane \(x_1 = -h\) in terms of electric potential \(\bar{\phi}\) and temperature \(\bar{T}\) at \(x_1 = h\).

**Part 2 of BCP I.1.3.** Next, we use (45), (46) along with (31), (34), (35) and (17) to determine the solution that satisfies the four remaining equations (25) joined to the four remaining boundary conditions
\[ t_1(h) = \bar{r}_1, \quad t_6(h) = \bar{r}_2, \quad u_1(-h) = \bar{u}_1, \quad u_2(-h) = \bar{u}_2. \quad (50) \]
Now by (17) and (34) we have $t_1 = \beta_1 T_1 e^{ah} + c_{11} U_{11} - \beta_1 T$ and $t_6 = c_{66} U_{21}$; hence by using (32)-(33) the boundary conditions (50) take the form

$$c_{11} U_{11} = \beta_1 T + \overline{r}_1 - \beta_1 T_1 e^{ah}, \quad c_{66} U_{21} = \overline{r}_2,$$

$$-U_{11} h + U_{12} = \overline{u}_1 - a^{-1} c_{11} \beta_1 T_1 e^{ah}, \quad -U_{21} h + U_{22} = \overline{u}_2,$$

with $T_1$ given by (42). On solving (51)-(52) for $U_{\alpha\beta}$ and substituting the resulting expressions in (34), (35) we are led to the complete solution of the second part of BCP I.1.3.

**Remark 4.1** We point out that in order to avoid growth as $h \to +\infty$ of the magnitude of the gradient

$$|\nabla u| + |\nabla T| + |\nabla \phi|$$

of any solution $(u, T, \phi)$ to BCP I.1.3, we assume

$$a := \frac{c_{44} \tilde{\omega}_1}{K(e_1^2 + c_{44} \varepsilon_{11})} > 0.$$  

(53)

In fact, note that by the equalities above, if $a < 0$, then $u_1, u_3, T, \phi \to \infty$ as $h \to \infty$.

4.1.4 On controllability in BVP I.1.3

By Eq.s (45), (46) we can deduce the following control property.

**Remark 4.2** Let $-h \leq x_1 < h$. For each choice of $\overline{u}_1, \overline{u}_2, \overline{v}_3, \overline{v}_1, \overline{v}_2, \overline{\phi}$, given any three quantities in $\{\overline{D}, \overline{I}_3, \overline{q}, \overline{T}\}$, the remaining quantity can be chosen to control either $T(x_1)$ or $\phi(x_1)$.

4.2 Problem BCP II.1.3

4.2.1 Statement of the problem

Here $P$, just as in BCP I.1.3, is bounded by the parallel planes $x_1 = \pm h$ each coated by an electrode, which is infinitesimally thin, so that all mechanical effects may be ignored. We seek solutions of the form (24) which when substituted in (19)-(23) give again Eq.s (25)-(27).

**BCP II.1.3 :** To find the solution of the form (24) to the field equations (25)-(27), which satisfies the ten boundary conditions
\[ T(h) = \overline{T}, \quad \phi(h) = \overline{\phi}, \quad t_1(h) = \overline{t}_1, \quad t_2(h) = \overline{t}_2, \quad t_3(h) = \overline{t}_3, \quad u_1(-h) = \overline{u}_1, \quad (i = 1, 2, 3), \quad \phi(-h) = \overline{\phi}_2, \quad -q_1(-h) = \overline{q}, \]  

(54)

with \( T, \overline{\phi}, \overline{t}_1, \overline{t}_2, \overline{t}_3, \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{\phi}_2 \) and \( \overline{q} \) assigned real constants.

4.2.2 General solution of BCP II.1.3

Insertion of (24) into the equilibrium field equations (19)-(23) gives Eq.s (25)–(27), whose general solution is expressed, as before, by Eq.s (31)-(33), where

\[ T_1, T_2, F_1, F_2, U_{a_1}, U_{a_2} \ (\alpha = 1, 2, 3) \]

are arbitrary smooth functions of \( \tau \).

4.2.3 Decomposition of BCP II.1.3

We solve BCP II.1.3 by separating it into two parts, described below.

Part 1 of BCP II.1.3.  We first determine the arbitrary constants in the general solution (31)–(33) so that the boundary conditions

\[ T(h) = \overline{T}, \quad \phi(h) = \overline{\phi}, \quad t_5(h) = \overline{t}_3, \quad u_3(-h) = \overline{u}_3, \quad \phi(-h) = \overline{\phi}_2, \quad -q_1(-h) = \overline{q} \]  

(55)

are satisfied. Note that by (15)2, (32) and (33) the last boundary condition becomes

\[ \kappa_1^E K T_1 - \kappa_1^E \phi_{,1} = -\kappa K^{-1} F_1 = \overline{q}. \]  

(56)

We have

\[ T_1 e^{ah} + T_2 = \overline{T}, \quad K T_1 e^{ah} + F_1 h + F_2 = \overline{\phi}, \]  

(57)

\[ e F_1 + c U_{31} = \overline{t}_3, \quad -e^{-1} e K T_1 e^{-ah} - U_{31} h + U_{32} = \overline{u}_3, \]  

(58)

\[ K T_1 e^{-ah} - F_1 h + F_2 = \overline{\phi}_2, \quad -\kappa K^{-1} F_1 = \overline{q}, \]  

(59)

and by solving the above system of equations for the unknowns \( (T_1, T_2, F_1, F_2, U_{31}, U_{32}) \) we find expressions for the latter in terms of the boundary data. In particular, we have

\[ T_1 = \frac{2h}{k(e^{ah} - e^{-ah})} \left[ \overline{q} + \frac{k'}{2h} \left( \overline{\phi} - \overline{\phi}_2 \right) \right], \quad T_2 = \overline{T} - e^{ah} T_1, \]  

(60)
\[ F_1 = -\frac{q}{k'}, \quad F_2 = \bar{\phi} + \frac{hK}{k} - Ke^{ah}T_1. \] (61)

Hence, by (32)–(33), the expressions of \( T \) and \( \phi \) in terms of the boundary data are

\[ T(x_1) = T + \frac{2h}{k} \left[ \bar{q} + \frac{k'}{2h} (\bar{\phi} - \bar{\phi}_2) \right] \frac{e^{ax_1} - e^{ah}}{e^{ah} - e^{-ah}}, \] (62)

\[ \phi(x_1) = \bar{\phi} + \frac{2h}{k'} \left[ \bar{q} + \frac{k'}{2h} (\bar{\phi} - \bar{\phi}_2) \right] \frac{e^{ax_1} - e^{ah}}{e^{ah} - e^{-ah}} + \frac{\bar{q}}{k'} (h - x_1), \] (63)

from which we deduce

\[ T(-h) = T - 2hk^{-1}\bar{q} - k'k^{-1}(\bar{\phi} - \bar{\phi}_2). \] (64)

**Part 2 of BCP II.1.3.** The remaining two equations, together with the appropriate boundary conditions, exactly coincide with BCP I.1.3. Hence, we can proceed as described at the end of Subsection 4.1.3.

**Remark 4.3** In order to avoid growth as \( h \to +\infty \) of the magnitude of the solution, we again assume (53).

### 4.2.4 On controllability of temperature

By Eq. (62) we can deduce the following control property.

**Remark 4.4** Let \(-h \leq x_1 < h\). For each choice of \( \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{t}_1, \bar{t}_2, \bar{t}_3 \), given any three quantities from \( \{ T, \bar{q}, \bar{\phi}, \bar{\phi}_2 \} \), the remaining quantity can be choosen to control \( T(x_1) \). In particular, if \( T \) and \( \bar{q} \) are assigned, then \( T(x_1) \) is controllable by \( \bar{\phi}_2 - \bar{\phi} \).

### 4.3 BCP I.3.3, plate perpendicular to the polarization direction

Now consider a plate occupied by the same above material but having the polarization direction \( x_3 \) perpendicular to the plane of the plate. The plate is coated by an infinitesimally thin electrode on the plane \( x_3 = h \), so that all its mechanical effects may be ignored. Solutions of the form

\[ T = T(x_3), \quad \phi = \phi(x_3), \quad u_i = u_i(x_3), \] (65)
satisfy (19)-(23) provided
\[ c_{44}u_{1,33} = 0, \quad c_{44}u_{2,33} = 0, \]  
\[ c_{33}u_{3,33} - \beta_3 T_{,3} + e_{33} \phi_{,33} = 0, \]  
\[ e_{33}u_{3,33} + \tilde{\omega}_3 T_{,3} - \varepsilon_{33} \phi_{,33} = 0, \quad -\kappa_{33} T_{,33} + \kappa_{33}^E \phi_{,33} = 0, \]  
which are a system included in the general case considered in the Appendix.

**BCP I.3.3**: To find the solution of the form (65) to the field equations (66)-(68), subject to the ten boundary conditions
\[
T(h) = T, \quad \phi(h) = \tilde{\phi}, \quad t_3(h) = \tilde{t}_1, \quad t_4(h) = \tilde{t}_2, \quad t_5(h) = \tilde{t}_3, \\
u_i(-h) = \bar{u}_i \quad (i = 1, 2, 3), \quad -D_3(-h) = \bar{D}, \quad -q_3(-h) = \bar{q}.
\]  

4.3.1 General solution of BCP I.3.3

In particular, on setting in (99)
\[ c = c_{33}, \quad e = e' = e_{33}, \quad \beta = \beta_3, \quad \omega = \tilde{\omega}_3, \quad \varepsilon = \varepsilon_{33}, \quad \kappa = \kappa_{33}, \quad \kappa' = \kappa_{33}^E, \]  
we obtain Eq.s (67)-(68). Then by Proposition 6.1 the general solution to (67)-(68) is
\[ u_1(x_3) = U_{11}x_3 + U_{12}, \quad u_2(x_3) = U_{21}x_3 + U_{22}, \]  
\[ u_3(x_3) = a^{-1} T_1 e^{ax_3} + U_{31}x_3 + U_{32} , \]  
\[ T(x_3) = T_1 e^{ax_3} + T_2, \]  
\[ \phi(x_3) = KT_1 e^{ax_3} + F_1 x_3 + F_2, \]  
where \( T_1, T_2, F_1, F_2, U_{a1}, U_{a2} \) \((\alpha = 1, 2, 3)\) are arbitrary smooth functions of \( \tau \) and the notation (70) and (101) is used. Moreover, the constant \( a \), given by
\[ a := \frac{\beta_3 e_{33} + c_{33} \tilde{\omega}_3}{K(e_{33}^2 + c_{33} \varepsilon_{33})}, \]
is supposed positive.
4.3.2 Decomposition of BCP I.3.3

We solve BCP I.3.3 by decomposing it into two parts. Part 1 of BCP I.3.3.

First, we note that Eqs (18), (72)–(74) and (101) imply

\[ D_3(x_3) = \omega T_1 e^{ax_3} + \omega T_2 + eU_{31} - \varepsilon F_1, \]  

(76)

and consequently, the solutions (72)–(74) meet the boundary conditions (36) when

\[ T_1 e^{ah} + T_2 = \bar{T}, \quad K T_1 e^{ah} + F_1 h + F_2 = \bar{T}, \]  

(77)

\[ -\beta T_2 + e F_1 + c U_{31} = \bar{T}_1, \quad a^{-1} c^{-1} V T_1 e^{-ah} - U_{31} h + U_{32} = \bar{u}_3, \]  

(78)

\[ -\omega T_2 - e U_{31} + \varepsilon F_1 = \bar{D}, \quad -\kappa K^{-1} F_1 = \bar{q}. \]  

(79)

By solving the above system of equations in the unknowns \((T_1, T_2, F_1, F_2, U_{31}, U_{32})\), we obtain

\[ T_2 = -\frac{\bar{T}_1}{\beta e + c\omega} + \frac{K k^{-1}(e^a + c\varepsilon)\bar{q}}{\beta e + c\omega e^{ah}}, \quad T_1 = e^{-ah}(\bar{T} - T_2), \]  

(80)

\[ F_1 = -\frac{\bar{q}}{k'}, \quad F_2 = \bar{T} - h F_1 - K e^{ah} T_1, \]  

(81)

\[ U_{31} = \frac{\omega \bar{T}_1 - \beta \bar{D} + k^{-1}(e\omega - \beta \varepsilon)\bar{q}}{\beta e + c\omega}, \quad U_{32} = \bar{u}_3 + h U_{31} - \frac{V}{ace^{ah}} T_1, \]  

(82)

which on substitution in the general solution (73)-(74) yields the solution to the first part of BCP I.1.3. In particular, we have

\[ \phi(x_3) = KT_1 \left(e^{ax_3} - e^{ah}\right) - k^{-1} \bar{q}(x_3 - h) + \bar{T}, \]  

(83)

\[ T(x_3) = T_2 \left(1 - e^{a(x_3 - h)}\right) + \bar{T} e^{a(x_3 - h)}, \]  

(84)

and thus, by (80) too, we have

\[ \phi(-h) = K \left[\bar{T} + \frac{\bar{T}_1 e^{ah} + \bar{D} + k^{-1}(e^a + c\varepsilon)\bar{q}}{(\beta e + c\omega)e^{ah}}\right](e^{-2ah} - 1) + \frac{2h}{k'} \bar{q} + \bar{T}, \]  

(85)

\[ T(-h) = \frac{\bar{T}_1 e^{ah} + \bar{D} + k^{-1}(e^a + c\varepsilon)\bar{q}}{(\beta e + c\omega)e^{ah}} \left(e^{-2ah} - 1\right) + e^{-2ah} \bar{T}. \]  

(86)
For \( 0 = t_1 = \overline{D} = \overline{q} \) we thus obtain
\[
\phi(-h) = K\left(e^{-2ah} - 1\right)T + \overline{\phi}, \quad T(-h) = e^{-2ah}T,
\]
which yields electric potential and temperature in the plane \( x_1 = -h \) in terms of electric potential and temperature \( \overline{\phi}, \overline{T} \) at \( x_1 = h \).

**Part 2 of BCP I.3.3.** The remaining two equations (66), subject to the remaining four boundary conditions (50), can be solved exactly as before in Subsection 4.1.3.

**Remark 4.5** We point out that in order to avoid growth as \( h \to +\infty \) of the magnitude of the gradient
\[
|\nabla u| + |\nabla T| + |\nabla \phi|,
\]
of any solution \((u, T, \phi)\) to BCP I.1.3, we assume
\[
a := \frac{\beta_3 e_{33} + c_{33}\tilde{\omega}_3}{K(e_{33}^2 + c_{33}e_{33})} > 0. \tag{88}
\]

### 4.4 On controllability in BCP I.3.3

By Eq.s (83), (84) we can deduce the following control property.

**Remark 4.6** Let \(-h \leq x_3 < h\). For each choice of \( \overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{t}_2, \overline{t}_3 \) and \( \overline{\phi} \), given any three quantities from \( \{\overline{t}_1, \overline{D}, \overline{q}, \overline{T}\} \), the fourth quantity can be chosen to control either \( T(x_3) \) or \( \phi(x_3) \).

### 4.5 BCP II.3.3, plate perpendicular to the polarization direction

Here \( \mathcal{P} \) (see BCP I.3.3) is bounded by the parallel planes \( x_3 = \pm h \) on which are coated two infinitesimally thin electrodes whose mechanical effects therefore may be ignored. We seek solutions of the form (65) which after substitution in (19)-(23) give Eq.s (66)-(68).

**BCP II.3.3:** To find the solution of the form (65) to the field equations (66)-(68), which satisfies the ten boundary conditions
\[
T(h) = \overline{T}, \quad \phi(h) = \overline{\phi}, \quad t_3(h) = \overline{t}_1, \quad t_4(h) = \overline{t}_2, \quad t_5(h) = \overline{t}_3, \\
u_i(-h) = \overline{u}_i \quad (i = 1, 2, 3), \quad \phi(-h) = \overline{\phi}_2, \quad -q_3(-h) = \overline{q}. \tag{89}
\]
4.5.1 General solution of BCP II.3.3

The equations corresponding to (66)-(68) are of the form (99) on setting
\[ c = c_{33}, \quad e = e' = e_{33}, \quad \beta = \beta_{3}, \quad \omega = \bar{\omega}_{3}, \quad \varepsilon = \varepsilon_{33}, \quad k = \kappa_{33}, \quad k' = \kappa_{33}^{E}, \tag{90} \]
so that by Proposition 6.1 the general solution to Eq.s (66)-(68) is given by Eq.s (71)-(74), where
\[ T_{1}, \quad T_{2}, \quad F_{1}, \quad F_{2}, \quad U_{a1}, \quad U_{a2} \quad (\alpha = 1, 2, 3) \]
are arbitrary smooth functions of \( \tau \) and we adopt the notation defined in Eq.s (70), (101).

4.5.2 Decomposition of BCP II.3.3

We solve BCP II.3.3 as follows:

**Part 1 of BCP II.3.3.** The general solution (72)–(74) to Eq.s (67)–(68) satisfies the six boundary conditions
\[
\begin{align*}
T(h) &= T, \quad \phi(h) = \bar{\phi}, \quad t_{3}(h) = \bar{t}_{1}, \\
u_{3}(-h) &= \bar{u}_{3}, \quad \phi(-h) = \bar{\phi}_{2}, \quad -q_{1}(-h) = \bar{q} \tag{91}
\end{align*}
\]
provided that
\[
\begin{align*}
T_{1} e^{ah} + T_{2} &= T, \quad K T_{1} e^{ah} + F_{1} h + F_{2} = \bar{\phi}, \tag{92} \\
-\beta T_{2} + 2 F_{1} + c U_{31} &= \bar{t}_{1}, \quad a^{-1} c^{-1} V T_{1} e^{-ah} - U_{31} h + U_{32} = \bar{u}_{3}, \tag{93} \\
K T_{1} e^{-ah} - F_{1} h + F_{2} &= \bar{\phi}_{2}, \quad -\kappa K^{-1} F_{1} = \bar{q}, \tag{94}
\end{align*}
\]
which can be solved for the unknowns
\[ (T_{1}, \quad T_{2}, \quad F_{1}, \quad F_{2}, \quad U_{31}, \quad U_{32}) \]
to give in particular the expressions
\[ T_{1} = \frac{2h}{k(e^{ah} - e^{-ah})} \left[ \bar{q} + \frac{k'}{2h} \left( \bar{\phi} - \bar{\phi}_{2} \right) \right], \tag{95} \]
and (60)$_{2}$, (61). Hence, (73) and (74) become
\[
T(x_{3}) = T + \frac{2h}{k} \left[ \bar{q} + \frac{k'}{2h} \left( \bar{\phi} - \bar{\phi}_{2} \right) \right] \frac{e^{ax_{3}} - e^{ah}}{e^{ah} - e^{-ah}}, \tag{96}
\]

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\[ \phi(x_3) = \frac{2h}{k'} \left[ q + \frac{k'}{2h} (\bar{\phi} - \bar{\phi}_2) \right] \frac{e^{ax_3} - e^{ah}}{e^{ah} - e^{-ah}} + \bar{\phi} + \frac{q}{k'} \left( h - x_3 \right), \tag{97} \]

and we conclude that
\[ T(-h) = T - 2hk^{-1}q - k'k^{-1}(\bar{\phi} - \bar{\phi}_2). \tag{98} \]

**Part 2 of BCP II.3.3.** The remaining two equations, subject to the appropriate boundary conditions, exactly coincide with the corresponding equations of BCP I.3.3. Hence we can solve them by the method of Subsection 4.1.3.

**Remark 4.7** In order to avoid growth as \( h \to +\infty \) in the magnitude of the gradient of the solution, we assume (88).

### 4.6 On controllability of temperature

By Eq. (96) we can deduce the following control property.

**Remark 4.8** Let \(-h \leq x_3 < h\). For each choice of \( \overline{\mathbf{v}} \) and \( \overline{\mathbf{u}} \), given any three quantities from \( \{T, q, \bar{\phi}, \bar{\phi}_2\} \), the remaining quantity can be chosen to control \( T(x_3) \). In particular, when \( T, q \) are assigned, \( T(x_3) \) is controllable by \( \bar{\phi} - \bar{\phi}_2 \).

### 5 Conclusions and perspectives

We have shown that, for a piezothermoelastic plate referred to a natural configuration, in the presence of a quasi-static incremental temperature given on one of its bounding faces, on the other bounding face either the electric potential or the temperature can be controlled by certain boundary data.

An aim of a future investigation could be to examine how these result generalize when the initial configuration of the plate is not a natural configuration, that is, when there is some initial mechanical, thermal and/or electric field.

### 6 Appendix

The following elementary result on first order differential equations is used.
Remark 6.1 If \( f = f(x) \) is a scalar function of the real variable \( x \) and \( a, b \in \mathbb{R} \), then the general solution of the linear first-order differential equation
\[
f' = a(f - b)
\]
is \( f = \gamma e^{ax} + b \), where \( \gamma \) is an arbitrary real constant.

Proposition 6.1 Let \( c, e, e', \beta, \omega, \varepsilon, k, k' \) be real scalars. Then the system of linear differential equations
\begin{align*}
cu_{,xx} - \beta T_{,x} + e' \phi_{,xx} &= 0 \\
eu_{,xx} + \omega T_{,x} - \varepsilon \phi_{,xx} &= 0 \\
-kT_{,xx} + k' \phi_{,xx} &= 0
\end{align*}
in the unknown scalar functions
\[
T = T(x), \quad \phi = \phi(x), \quad u = u(x),
\]
of the real variable \( x \), has the general solution
\[
T(x) = T_1 e^{ax} + T_2 \\
\phi(x) = KT_1 e^{ax} + F_1 x + F_2 \\
u = a^{-1}c^{-1}VT_1 e^{ax} + U_1 x + U_2
\]
where
\[
(T_1, T_2, F_1, F_2, U_1, U_2) \in \mathbb{R}^6
\]
are arbitrary and
\[
K := k/k', \quad A := \beta e + c\omega, \quad B := ee' + c\varepsilon, \quad a := AK^{-1}B^{-1}, \quad V := \beta - Kae'.
\]

Proof. Equation (99)_3 yields \( \phi_{,xx} = KT_{,xx} \), thus
\[
\phi_{,x} = KT_{,x} + F_1, \quad F_1 \in \mathbb{R},
\]
and Eq.s (99)_1, 2 become
\[
u_{,xx} = c^{-1}(\beta T_{,x} - e'KT_{,xx}), \quad u_{,xx} = e^{-1}(-\omega T_{,x} + \varepsilon KT_{,xx}).
\]
By eliminating \( u_{,xx} \) from these two equalities, we obtain the second-order equation
\[
T_{,xx} = aT_{,x},
\]
with \( a \) defined in (101); consequently \( T_x = a(T - T_2) \) where \( T_2 \) is an arbitrary constant. By Remark 6.1 the latter equation has the general solution

\[
T(x) = T_1 e^{ax} + T_2 \quad (T_1, T_2) \in \mathbb{R}^2.
\]

(104)

which by substitution in (102) enables us to conclude that (100) holds. Lastly, insertion of the expressions for \( T \) and \( \phi \) into (103) yields \( u_{,xx} = c^{-1}T_1 V e^{ax} \) where \( V := \beta - Ke^{a}a. \) Hence by integration we obtain \( u_{,x} = c^{-1}T_1 V e^{ax} + U_1, \) which yields (100)3. ♦

7 Acknowledgments

The author would like to thank Professor R. Knops for his discussion and suggestions on the present paper.

References


