Implicit Integration Algorithm for an Enhanced Generalized Plasticity Constitutive Model of Partially Saturated Soils

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Abstract The objective of this paper is to investigate the stress update algorithm and the tangent operator for an enhanced generalized plasticity model developed recently for partially saturated soils, where on top of the hydrostatic and deviatoric components of the (effective) stress tensor suction has to be considered as an independent variable. The soil model used for the applications can be derived from a potential energy function and thus present thermodynamically consistent in the form of the effective stress tensor. The Galerkin method is used to obtain a weak form of the governing equations and a “stress-suction coupling matrix” is derived in the discretization in space by finite element method. The implicit integration algorithm is incorporated in a code for partially saturated soil dynamics. The framework of generalized plasticity is presented and the numerical results of subsidence of reservoir near Venice are presented.

Key Words Generalized Plasticity, Implicit Integration Algorithm, Return Mapping Algorithm, Consistent Tangent Stiffness Matrix, Partially Saturated Soils

1. Introduction

Partially saturated soil behaviour has received recently much attention and several constitutive models have been proposed to describe its behaviour. These models are based on several choices of the stress tensors and it appears from literature that the effective stress as defined originally by Lewis and Schrefler (1982) in conjunction with suction is the most popular form. In fact, this pair of stresses is used by Jonmi and Di Prisco (1994), Bolzon et al. (1996), Houlsby (1997), Zienkiewicz et al. (1999), Hofstetter et al. (1999), Gallipoli et al. (2003), Sheng et al. (2003a,b), Eberhardsteiner et al. (2003), Borja (2004), Tamagnini and Pastor (2004) and Sheng et al. (2004), Ehlers et al. (2004). With this choice the change from fully saturated states to partially saturated ones is straightforward and also the possible hysteresis of the capillary pressure–saturation relationship can be accounted for, see Chateau and Dormieu (2002). This relationship is a keystone in partially saturated soil mechanics, (see Lewis and Schrefler, 1998). The above and other forms of the solid stress are discussed in more detail in Gray and Schrefler[to appear].

Recently an enhanced version of the constitutive model BSZ proposed by Bolzon et al. (1996) has been developed (Santagiuliana and Schrefler(2006)). This enhanced model is referred to as
BSZe model, to distinguish it from the original BSZ version. The BSZ model already considers suction effects on the stiffness and is capable to simulate swelling and subsequent collapse of a geomaterial when suction decreases. Even if suction was considered as second stress variable, it did not appear explicitly in the model formulation. In the enhanced model suction appears explicitly as stress variable and the dependence of the plastic rate strain on suction is taken into account. Jommi and Di Prisco (1994) and Tamagnini and Pastor (2004) have proposed such a dependence by defining the total strain rate as a sum of three components: the elastic strain tensor, the plastic strain tensor coupled with the effective stress tensor, and the strain tensor coupled with suction. This dependence of the plastic strain rate on suction takes into account irreversible deformation during cyclic drying and wetting, when the suction increases or decreases, until the structural collapse, while the effective stress remains constant.

The BSZ model is thermodynamically consistent as shown by Santagatiuliana and Schrefler (2006) and is entirely determined by the knowledge of two scalar potentials, i.e. the Helmholtz free energy and the dissipation function. These potential functions are defined for the case of associative and non-associative plasticity. This last one refers to “frictional materials”, Collins and Houlsby (1997).

It is recalled that the BSZ model was developed starting from a generalized plasticity constitutive model proposed by Pastor and Zienkiewicz (Pastor et al. (1990)) to predict the basic phenomena encountered in dynamic loading of fully saturated porous media such as the accumulation of plastic strain and pore pressure build-up during the loading process. The generalized plasticity does not require yield and plastic potential surfaces to be explicitly defined.

It is well known that in classical plasticity the hardening modulus is fully determined by the enforcement of the consistency condition (De Borst and Heeres (2000)), while in generalized plasticity it can be chosen freely. This allows to select different expressions of this modulus for loading and unloading, which is particularly useful for cyclic loading. Thus, the model is in principle capable to represent the mechanical behaviour of both dense and loose sands under quasi-static and dynamic loadings and can be used in several situations such as the analysis of static liquefaction of very loose sands, instability phenomena during undrained loading of medium-dense specimens, material softening and strain localization of sands, etc. The handling of the nonlinearities and of the stress updates are sometimes based on a simple integration method, namely the forward Euler algorithm. This may produce very low accuracy and convergence.

In classical plasticity these problems were solved by the introduction of an implicit algorithm for the integration of the elasto-plastic rate equations and by the use of tangent operators consistent with this integration algorithm (Simo and Taylor (1986)). Recently, De Borst and Heeres (2000) have recognized that these concepts can be extended to non-classical plasticity models. Zhang et al. (2001) developed for the BSZ model an implicit stress update algorithm together with a tangent operator consistent with it. These ideas are here extended to the case of the enhanced generalized plasticity model for partially saturated soils, where stresses and suction have to be considered explicitly as stress variables.

A few basic facts are now recalled.

As shown by Houlsby (1979) the effective stress in the form

\[
\sigma'_{ij} = \sigma_{ij} + \left[ S_w p'^w + S_g p'^s \right] \delta_{ij}
\]

and suction, also called capillary pressure

\[
s = p'^s = p'^s - p'^w
\]

scaled by porosity, \(n\), (modified suction, \(ns\)) are work-conjugate respectively with the solid
strain $\varepsilon_{ij}$ and the degree of saturation $S_w$. The stress variables are not independent: the capillary pressure is contained in the effective stress because of the saturation constraint (the saturations sum to one). This means that changes in suction may produce changes in effective stress and vice versa. In the above equations $\sigma_{ij}$ is the total stress tensor, $\sigma'_{ij}$ is the “effective” stress responsible for deformations, $\delta_{ij}$ the Kronecker symbol and $p^w$ and $p^g$ are the water and gas pressures, respectively. Traction stresses and compressive pressures are here assumed as positive.

Usually strains (through displacements) and the capillary pressure are obtained from the solution of a boundary value problem, while the saturation is obtained from the capillary pressure through the already mentioned capillary pressure-saturation relationship. This relationship may present hysteresis.

The outline of the paper is the following. First the principal concepts of generalized plasticity are briefly recalled. Successively the enhanced Bolzon-Schrefler-Zienkiewicz generalized plasticity model for partially saturated soil is described, starting directly from the dissipation function and the Helmholtz free energy. Then the stress update algorithm and the consistent operator are developed along the lines of de Borst and Heeres (2000) and Zhang et al. (2001). The implementation of the return-mapping algorithm is given and the method of general closest point projection iteration is selected for the analysis of partially saturated soils. Since for the examples we need the solution of a boundary value problem, the problem and its solution is shown, following Schrefler and Scotta (2001). It is important to point out that here we need a mass balance equation for the gas phase (usually, but not always air) in addition to that of the water phase. New is the appearance of a “stress-suction coupling matrix” in the momentum balance equation for the mixture solid-water-air due to the tangent operator adopted. Several examples which show the advantage of the proposed algorithm conclude the paper.

Finally, in order to demonstrate the applicability of the proposed model and algorithm, the analysis of reservoir subsidence near Venice is presented using the enhanced Pastor-Zienkiewicz model.

Finally, an experimental validation of this model is presented. Theoretical predictions of the enhanced model in drained triaxial tests are made and compared with experimental observations for the saturated and unsaturated case. The chosen material is Sion slime (see Geiser (1999)).

4.2 Generalized plasticity theory for saturated soils

For the solid phase, as far as the constitutive laws is concerned, there are many choices, including linear and non linear elasticity, generalised plasticity, in particular the Pastor-Zienkiewicz model for cyclically loaded sand and damage mechanics.

With the previous assumptions and if only small displacements are considered, the stress-strain relationship for the porous medium can be written in the general incremental form

\[
d\sigma_{ij}^* = D_{ijkl} \left( d\varepsilon_{kl} - d\varepsilon_{kl}^0 \right)
\]

(3)

where $D_{ijkl}$ is the fourth order tangential stiffness tensor of the material (generally variable with the adopted state variables and the direction of their increment in a material non-linear analysis), $d\varepsilon_{ij}$ is the second order tensor describing the total strain increment and $d\varepsilon_{kl}^0$ is the fraction of the strain increment not induced by the effective stress (thermal strain, viscous strain,
shrinkage, etc.). Equation (3) must be adequately modified if large displacements are considered (Meroi et al. (1995)).

For plastic analysis the constitutive tensors $D^{ep}$ and $C^{ep}$ (inverse of $D^{ep}$) should be such that all materials symmetries are preserved and will generally depend on the current state variables and on the direction of loading. We assume that the material is achronic, i.e., its behaviour is independent of the rate of loading, and with the definition of the stress $\sigma$, the direction of loading $\gamma$ and the state variables $\alpha$, we can write

$$C^{ep} = C^{ep}(\gamma, \sigma, \alpha), \quad D^{ep} = D^{ep}(\gamma, \sigma, \alpha)$$

in which

$$\gamma_{ij} = \frac{d\sigma_{ij}}{\sqrt{d\sigma_{kk}d\sigma_{ll}}}$$

It is assumed that the deformation of the material can be considered as the result of deformations produced by $M$ separate mechanisms, all of these subjected to the same state of stress. The strain increment can thus be written as

$$d\varepsilon_{ij} = \sum_{m=1}^{M} C^{ep(m)}_{ijkl} d\sigma_{kl} = C^{ep}_{ijkl} d\sigma_{kl}$$

and

$$C^{ep}_{ijkl} = \sum_{m=1}^{M} C^{ep(m)}_{ijkl}$$

Furthermore, a direction vector $n^{(m)}$ is postulated in the stress space discriminating between loading and unloading for each mechanism. The loading and unloading are respectively defined as

**Loading:**

$$d\sigma_{ij} n^{(m)}_{ij} > 0$$

**Unloading:**

$$d\sigma_{ij} n^{(m)}_{ij} < 0$$

The difference with classical plasticity is that the surfaces $f^{(m)}$ need never be defined explicitly. The limit case is called neutral loading and is defined by

$$d\sigma_{ij} n^{(m)}_{ij} = 0$$

The above definitions hold for hardening and perfecting plastic materials and will be generalized to account for softening later.

Continuity between loading and unloading processes requires that constitutive tensors $C^{ep L(m)}$ (loading) and $C^{ep U(m)}$ (unloading) for a single mechanism $(m)$ have the form
\[ C_{ijkl}^{epL(m)} = C_{ijkl}^{e(m)} + \frac{1}{H_L} n_{ijgU}^{(m)} n_{kl}^{(m)} \]
\[ C_{ijkl}^{epU(m)} = C_{ijkl}^{e(m)} + \frac{1}{H_U} n_{ijgU}^{(m)} n_{kl}^{(m)} \]

(11)

where \( n_{gL/U} \) are arbitrary unit tensors which determine the flow direction. \( H_{L/U} \) are plastic moduli corresponding to loading and unloading.

For neutral loading case, equation (10) holds as well as

\[
\begin{align*}
  d\varepsilon_{ij}^{L(m)} &= C_{ijkl}^{e(m)} d\sigma_{kl} \\
  d\varepsilon_{ij}^{U(m)} &= C_{ijkl}^{e(m)} d\sigma_{kl}
\end{align*}
\]

(12)

It is clear that both laws predict the same strain increment under neutral loading in which both expressions are valid and hence non-uniqueness is avoided.

Material behaviour under neutral loading is reversible and it can therefore be regarded as elastic, as, given an infinitesimal cycle \( +d\sigma_{ij}, -d\sigma_{ij} \), we will have

\[ d\varepsilon_{ij}^{(m)} = C_{ijkl}^{epL(m)} d\sigma_{kl} + C_{ijkl}^{epU(m)} (-d\sigma_{kl}) = 0 \]

(13)

The increment of strain caused by the mechanism (m) due to \( d\sigma_{ij} \) can be assumed to have two components, an elastic one and a plastic one, and it has the form

\[ d\varepsilon_{ij}^{(m)} = d\varepsilon_{ij}^{e(m)} + d\varepsilon_{ij}^{p(m)} \]

(14)

where

\[
\begin{align*}
  d\varepsilon_{ij}^{e(m)} &= C_{ijkl}^{e(m)} d\sigma_{kl} \\
  d\varepsilon_{ij}^{p(m)} &= \frac{1}{H_{L/U}} \left[ n_{ijgL/U}^{(m)} n_{kl}^{(m)} \right] d\sigma_{kl}
\end{align*}
\]

(15)

Irreversible-plastic deformations have been introduced without the need for specifying yield or plastic potential surfaces. All that is necessary to specify the behavior for mechanism (m) are two scalar functions \( H_{L/U}^{(m)} \) and the directions \( n_{ijgL/U}^{(m)} \) and \( n_{ij}^{(m)} \).

To account for the softening behavior of the material, e.g. in localisation analysis, the definitions for loading and unloading have to be modified. They can be written as

\[
\begin{align*}
  n_{ij}^{(m)} : d\sigma_{ij}^{(m)} > 0 & \quad \text{loading} \\
  n_{ij}^{(m)} : d\sigma_{ij}^{(m)} = 0 & \quad \text{neutral loading} \\
  n_{ij}^{(m)} : d\sigma_{ij}^{(m)} < 0 & \quad \text{unloading}
\end{align*}
\]

(16)

where \( d\sigma_{ij}^{(m)} \) is given by

\[ d\sigma_{ij}^{(m)} = C_{ijkl}^{e(m)-1} d\varepsilon_{ij}^{(m)} \]

(17)
It is easily verified that for positive-definite $C_{ijkl}^\epsilon$, the two definitions above are identical when no softening exists.

Finally, total increment of strain is given by the summation of all mechanisms, i.e.

$$d\varepsilon_{ij} = \sum_{m=1}^{M} C_{ijkl}^{\epsilon(m)} d\sigma_{kl} + \sum_{m=1}^{M} \frac{1}{H_{L/U}^m} \left[ n_{ijkl}^{(m)} n_{im}^{(m)} \right] d\sigma_{ik}$$  \hspace{1cm} (18)

or

$$d\varepsilon_{ij} = C_{ijkl,L,U}^{\epsilon} d\sigma_{kl}$$  \hspace{1cm} (19)

where

$$C_{ijkl,L,U}^{\epsilon} = \sum_{m=1}^{M} C_{ijkl}^{\epsilon(m)} + \sum_{m=1}^{M} \frac{1}{H_{L/U}^m} \left[ n_{ijkl}^{(m)} n_{ij}^{(m)} \right]$$  \hspace{1cm} (20)

Inversion of $C_{ijkl,L,U}^{\epsilon(m)}$ will give the stiffness tensor $D_{ijkl}^{\epsilon}$ as

$$D_{ijkl}^{\epsilon(m)} = D_{ijkl}^{\epsilon} - (D_{ijmn}^{\epsilon} P_{mn}^{(m)}) B^{(m)} (D_{ijkl}^{\epsilon'}) Q_{ijkl}^{(m)}$$  \hspace{1cm} (21)

where

$$Q_{ij}^{(m)} = n_{ij}^{(m)}, \quad P_{ij}^{(m)} = n_{ij}^{(m)} L_{ij}, \quad B^{(m)} = H^{(m)} + (P_{kl}^{(m)} D_{klmn}^{\epsilon(m)} Q_{mn}^{(m)})$$

For the simple case of $M=1$, expression (21) reduces to

$$D_{ijkl}^{\epsilon} = D_{ijkl}^{\epsilon'} - \frac{(D_{ijmn}^{\epsilon} P_{mn}) (D_{ijkl}^{\epsilon'} Q_{ijkl})}{H + P_{ij} D_{ijmn}^{\epsilon} Q_{mn}}$$  \hspace{1cm} (22)

### 4.3 Enhanced generalized plasticity sand model for partially saturated materials

The enhanced model used for partially saturated geomaterials has been developed by Santagiuliana, Schrefler (2006) as an extension of the Pastor-Zienkiewicz and Bolzon-Schrefler-Zienkiewicz generalised plasticity models for fully and partially saturated (Pastor et al. (1990), Pastor and Zienkiewicz (1985, 1986) and Bolzon et al. (1996)). To consider the mechanics of partially saturated porous media, it is assumed that the stress variables are the soil suction $s$ and Bishop’s effective stress $\sigma^e_{ij}$, as shown in equation (1),

$$\sigma^e_{ij} = \sigma_{ij} + \left[ S_w p^w + (1 - S_w) p^s \right] \delta_{ij} = \sigma_{ij} + \left[ p^s - S_w s \right] \delta_{ij}$$  \hspace{1cm} (24)

with soil suction $s$ defined in equation (2).

In equation (24), parameter $S_w$ represents a phenomenological measure of the capillary effects, through its experimental relationship with suction. Bishop’s stress definition recovers the
Terzaghi’s effective stress definition, usually assumed in fully saturated soil mechanics, when saturation equals one, hence the consistency condition between stress measures is guaranteed. From Equation (24) it results that changes in Bishop’s stress may be induced by changes in the total stress but also by changes in the gas pressure, suction and water saturation. The elasto-plastic model is now defined in the space \((p', q, s)\) of the mean stress \(p'\)

\[
p' = -\frac{\sigma''}{3} = p + S_u p_v + (1 - S_u) p_g
\]

the deviatoric stress \(q\) (both invariants of Bishop’s stress)

\[
q = \sqrt{3J_2}, \quad J_2 = \frac{1}{2} \sigma'_{ij}\sigma'_{ji}
\]

and the suction \(s\).

The plastic direction vectors are respectively defined as

\[
n^p = -M_f (1 + \frac{1}{c})(1 - \frac{(cp'^{c+1} + 1)p'}{p_c^c}), \quad n^q = 1,
\]

\[
n^p = -M_f (1 + c) \frac{(RI)p'^c}{p_c^c},
\]

\[
n^p = -M_g (1 + \frac{1}{c})(1 - \frac{(cp'^{c+1} + 1)p'}{p_c^c}), \quad n^q = 1,
\]

\[
n^p = -M_g (1 + c) \frac{(RI)p'^c}{p_c^c}
\]

where \(c, M_g, M_f\) and \(p_f\) are material parameters and \(p_f\) is a suction dependent function.

In the case of unloading, irreversible strains are of a contractive nature. Therefore the unloading plastic flow direction vector \(n_{gU}\) is defined as

\[
n_{gU} = (n^p_{gU}, n^q_{gU}, n^s_{gU})
\]

and

\[
n_{gU}^p = -n_{gU}, \quad n_{gU}^q = n_{gU}^s, \quad n_{gU}^s = n_{gU}^s
\]

The direction vector discriminating between loading and unloading is conveniently defined in a form similar to the plastic flow direction.

Just as we discussed before, in the critical state models the strain increment can be decomposed into an elastic part and an elastoplastic one, and the volumetric strain work-conjugate with \(p'\) is defined as

\[
\varepsilon_v^{(m)} = -tr(\varepsilon_q^{(m)})
\]

The deviatoric strain work conjugate with \(q\) is defined as
\[ \varepsilon_s^{(m)} = \frac{2}{3} \sqrt{\text{dev}(\varepsilon_{ij}^{(m)}) \text{dev}(\varepsilon_{ij}^{(m)})} \]  

(32)

All these components of strain are positive in compression.

To have the thermodynamic formulation of a model we can note (Collins and Houlsby, 1997), that in an isothermal process the rate of work input per unit volume is equal to the sum of the rate of change of free energy \( \dot{A} \) and the dissipation \( D \).

\[ \dot{W} = \dot{A} + D \]  

(33)

The former represents the rate of change of recoverable, not necessarily elastic, energy, while the dissipation is the rate at which energy is dissipated. This shows that the two fundamental thermodynamic functions, i.e. the free energy potential and the dissipation function, uniquely define the constitutive model, and all the necessary equations can be derived from them.

The effective stress \( \sigma' \) and the modified suction (scaled by the porosity) \( n_s \) are work conjugate with the rate of soil skeleton strain \( \dot{\varepsilon} \) and the rate of degree of saturation \( \dot{S}_w \), respectively. The suction plays the role of stress, the degree of water saturation is a strain-like variable.

Following Houlsby (1997), Tamagnini and Pastor (2004) and neglecting the mechanical dissipation associated with fluid flows and the air compressibility, the rate of work input per unit volume of unsaturated soil is

\[ \dot{W} = [\sigma_{ij} + p^s \delta_{ij} - S_w (p^s - p^w) \delta_{ij}] \dot{\varepsilon}_{ij} - n (p^s - p^w) \dot{S}_w \]  

(34)

\[ \dot{W} = \sigma'_{ij} \dot{\varepsilon}_{ij} - n_s \dot{S}_w \]  

(35)

The Helmholtz free energy is defined in Schrefler (2002) for the phases and interfaces that compose the porous material. The soil is here investigated in terms of effective stress and solid strains, therefore we use the Helmholtz free energy associated with the solid skeleton for unsaturated soils and this is supposed to depend on the soil skeleton strains \( \varepsilon_{ij} \), on kinematic strain-like internal variable \( \alpha_{ij} \) and on degree of water saturation

\[ A' (\varepsilon_{ij}, \alpha_{ij}, S_w) \]  

(36)

The kinematic strain-like internal variable \( \alpha_{ij} \) represents the internal changes of the material, e.g. plastic strain or damage, that can sometimes be measured but they cannot be controlled from the outside.

The Helmholtz free energy, for an isothermal process, can be rewritten as

\[ A' = A_1 (\varepsilon_{ij}^p, S_w^p) + A_2 (\alpha_{ij}, S_w^p) \]  

(37)

where the second component \( A_2 (\varepsilon_{ij}^p, S_w^p) \) is the part of the Helmholtz free energy that depends on plastic strains and on the saturation degree.

For a general isothermal process the rate of plastic component of work is
\[ \dot{W}^p = \dot{A}_2 + D \]  

and can be expressed in triaxial stress state terms as suggested by Sheng et al. (2004)

\[ \dot{W}^p = p^1 \dot{\varepsilon}_v^p + q \dot{\varepsilon}_t^p - nsS_u^p \]  

The enhanced BSZ model, differently from the modified Cam-Clay model, can be generated without the need to introduce a back stress (Collins and Houlsby, 1997)) and so all the plastic work due to the first two terms of Equation (39) is dissipated.

The last term in Equation (39) is only relevant to suction increase and suction decrease, the movement of the yield surface \( f \) alone does not contribute to \( S_u^p \).

This term is a state function depending only on the current saturation degree value, hence it is integrable and gives zero in a closed loop of the degree of water saturation. According to Equation (38), it can be considered as the second part of the Helmholtz free energy:

\[ \dot{A}_2 = -nsS_u^p. \]  

This equation indicates that suction increase and decrease yield surfaces contribute to the plastic work and the plastic work product is recoverable. During the loading path in unsaturated conditions, the stored energy is recovered by wetting, when the forces due to capillarity are removed; the resulting behaviour is called “structural collapse” (Tamagnini and Pastor, 2004).

In the associative formulation of the BSZ_{e} model, \( f \equiv g \) and therefore \( M^f_f \equiv M^s_s \). We consider the case of associative plasticity.

The dissipation function is generally obtained from Equations (38), (39) and (40) as Equation (41).

\[ D = \dot{W}^p - \dot{A}_2 = p^1 \dot{\varepsilon}_v^p + q \dot{\varepsilon}_t^p \]  

For the BSZ_{e} model it is here defined, in associative plasticity, as a function of the volumetric and deviatoric plastic strain rate

\[ D = M^f_f \dot{\varepsilon}_v^p (1 + c) p^f_f \left( \frac{1 + c}{1 + c} M^f_f \dot{\varepsilon}_t^p + c \dot{\varepsilon}_t^p \right)^{\frac{1}{1 + c}} = M^f_f \dot{\varepsilon}_v^p (1 + c) p^f_f \left( \frac{p^1}{p^f_f} \right)^c \]  

Note that the dissipation, Equation (42) is a homogeneous function of degree one in the plastic strain increments, therefore it coincides with the dissipation potential function for Euler’s Theorem.

Hence the stresses \( p^1 \) and \( q \) can be derived from the function \( D \):

\[ p^1 = \frac{\partial D}{\partial \dot{\varepsilon}_v^p} = p^f_f \left( \frac{c}{1 + c} + \frac{\dot{\varepsilon}_t^p}{\dot{\varepsilon}_v^p (1 + c) M^f_f} \right)^{\frac{1}{c}} \]  

Note that the dissipation, Equation (42) is a homogeneous function of degree one in the plastic strain increments, therefore it coincides with the dissipation potential function for Euler’s Theorem.

Hence the stresses \( p^1 \) and \( q \) can be derived from the function \( D \):
The associated flow rule for the volumetric and deviatoric part of plastic strain is:

\[
\dot{\varepsilon}_v^p = \dot{\lambda}(1+c)(M_f - \eta), \quad \dot{\varepsilon}_s^p = \dot{\lambda}
\]  

(45)

where \( \dot{\lambda} \) is the plastic multiplier. By combining the two Equations (45) and the expression of \( p' \) and \( q \), (43) and (44), we obtain

\[
f = q - M_f p' \left(1 + \frac{1}{c}\right) \left[1 - \left(\frac{p'}{p_f}\right)^{c}\right]
\]

(46)

equation that describes the yield function \( f \) of the BSZ model, obtained also by integration of Equations (27). This equation coincides with the equation of the potential surface:

\[
g = q - M_g p' \left(1 + \frac{1}{c}\right) \left[1 - \left(\frac{p'}{p_g}\right)^{c}\right],
\]

(47)

obtained by integration of Equations (28), because of associative plasticity. Differentiating, instead, the Helmoltz-free energy

\[
A' = \frac{k}{v_0} \varepsilon_v^{\varepsilon_v} + \frac{(\varepsilon_v^\varepsilon_v)^2}{2} G - nsS_w,
\]

(48)

where \( k \) is a constant independent on suction and \( v_0 \) is the specific volume, we can also obtain the stress

\[
p' = \frac{\partial A'}{\partial \varepsilon_v^\varepsilon_v}, \quad q = \frac{\partial A'}{\partial \varepsilon_s^\varepsilon_s}
\]

(49)

and the suction, as already indicated in Equation (40). Furthermore, from the Gibbs free-energy expression, obtained from the Helmholtz free-energy function, by a Legendre transformation,

\[
G' = \frac{k}{v_0} p'(\ln p' - 1) + \frac{q^2}{2G} + p' e_v^\varepsilon_v + q e_s^\varepsilon_s - nsS_w
\]

(50)

we can obtain the elastic deformations, because it must be:
\[ \frac{\partial G^p}{\partial p'} = \varepsilon_v \quad \text{and} \quad \frac{\partial G^q}{\partial q} = \varepsilon_s \]  

and particularly

\[ \frac{\partial G'_1}{\partial p'} = \varepsilon'_v \quad \text{and} \quad \frac{\partial G'_1}{\partial q} = \varepsilon'_s \]  

where

\[ G_i(p',q) = \frac{k}{v_0} p' (\ln p' - 1) + \frac{q^2}{2G}, \]  

\[ \varepsilon^e_v = \frac{k}{v_0} \ln p' \quad \text{elastic part of volumetric strain} \]  

\[ \varepsilon^e_s = \frac{q}{G} \quad \text{elastic part of shear strain} \]  

where \( G \) is the shear modulus.

When \( M_f \neq M_g \), the flow rule is of non-associative type, so the normal to the yield surface is different from the normal to the plastic potential surface, the expressions of \( p' \) and \( q \) are:

\[ p' = p_f \left\{ \left( \frac{M_f (1+c)^2 - c(1+c)M_g + c \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_p}}{M_f (1+c)^2} \right) \right\}^{\frac{1}{2}} \]  

\[ q = p_f \left\{ \left( \frac{M_f (1+c)^2 - c(1+c)M_g + c \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_p}}{M_f (1+c)^2} \right) \right\}^{\frac{1}{2}} \left( M_g - \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_p (1+c)} \right) \]  

For the BSZ\(_c\) model the dissipation function for non associative plasticity is defined as:

\[ D = p_f \left\{ \left( \frac{M_f (1+c)^2 - c(1+c)M_g + c \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_p}}{M_f (1+c)^2} \right) \right\}^{\frac{1}{2}} \left[ \left( M_g - \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_p (1+c)} \right) \dot{\varepsilon}_p + \dot{\varepsilon}_v \right] \]
This dissipation function can also be written as

\[
D = \dot{\varepsilon}_v^p (1 + c) \left( M_g - M_f + M_f \left( \frac{p'}{p_f} \right)^\epsilon \right)
\]  

(59)

by using

\[
M_f \left( 1 - \left( \frac{p'}{p_f} \right)^\epsilon \right) = M_g \left( 1 - \left( \frac{p'}{p_g} \right)^\epsilon \right)
\]  

(61)

This last equation is the condition derived by considering that we have plastic flow only when the yield function \( f = 0 \) and the potential function \( g = 0 \).

In the case of non-associative flow, the derivation of the stress tensors \( p' \) and \( q \) from the dissipation function \( D \), needs the application of Lagrangian multiplier method (Lippmann, 1972) as it is shown in Santagiuliana and Schrefler (2006).

The non-associated flow rule is:

\[
\dot{\varepsilon}_v^p = \dot{\lambda}(1 + c)(M_g - \eta) , \quad \dot{\varepsilon}_v^p = \dot{\lambda}
\]  

(62)

By combining the two Equations (62) and the expression of \( p' \) and \( q \), obtained by the application of Lagrangian multiplier method, we derive the Equation (46) that describes the yield function \( f \) of the BSZ model.

The Gibbs and Helmholtz free energies are the same as in the case of associative plasticity.

Both the dissipation functions for associative and non-associative plasticity are shown in Santagiuliana and Schrefler (2006) to be always non-negative and hence thermodynamically consistent.

From Equation (54), during elastic unloading, with constant suction, the volumetric strain increment corresponds to the elastic strain increment which depends on stress increment

\[
d\varepsilon_v^{(e)} = \frac{k}{v_0} \frac{\text{d} p'}{p'}
\]  

(63)

Further during plastic loading with constant suction the volumetric strain increment is assumed to be composed of an elastic and a plastic part
where $\lambda$, the soil compressibility, is a constant dependent on suction.

From Equation (55), the elastic deviatoric strain increment is

$$d\varepsilon_{v}^{(m)} = \frac{1}{G} \, dq$$

(65)

The effect of suction is here taken into account in the constitutive equation for partially saturated soils through the introduction of a term that depends on suction as in the constitutive formulations of models for unsaturated soils. The plastic part of the rate of strains in unsaturated soil is obtained from the sum of two terms

$$\dot{\varepsilon}_{ij}^{(m)} = \frac{1}{H} \dot{\sigma}_{kl} n_{ij} n_{kl} + \frac{1}{H_b} \dot{\lambda} n_{ij}$$

(66)

where $H$ is the plastic modulus which depends on the material characteristics and explicit expressions for clays and sands are given by Pastor et al. (1990) and Zienkiewicz et al. (1990a). $H_b$ is the hardening modulus related with suction.

The first term in equation (66) is the hardening law valid for saturated soil. The second term refers only to unsaturated soil, dependent on suction and impose the dependence of plastic strain also on changes in suction during wetting.

This dependence of the soil stiffness on the suction complies with laboratory experiments in which unsaturated specimens exhibit a lower volume change than saturated ones when subjected to the same increment of vertical stress. Further, when the unsaturated samples are soaked and hence saturated, the soil exhibits a significant volumetric strain under constant stress.

The enlargement of the region of the elastic behaviour due to stress increments depends on plastic hardening. The hierarchical structure of generalised plasticity allows to choose a loading hardening parameter of the type

$$H_L = H_o p' H_s H_v (H_v + H_s) H_{dm}$$

(67)

In Equation (67) the term $H_0$ depends on the material characteristics in fully saturated conditions whereas $H_w$ is related to partially saturated behaviour. These coefficients are respectively assumed as

$$H_0 = \frac{1 + e_0}{\lambda (0) - \kappa (0)}, \quad H_w = 1 + as$$

(68)

being $e_0$ the initial void ratio. $H_f$ is the stress path dependent term. $H_s$ is the deviatoric strain hardening function which will go to zero as deviatoric deformation increases. $H_f (H_s + H_v)$ depends on deviatoric strain. When both $H_f$ and $H_s$ approach zero, the soil will fail and the residual conditions will take place on the critical state line (see also Simoni and Schrefler (2001))
and Pastor et al. (1990)). The parameters in (67) can be defined as

\[ H_f = \left(1 - \frac{\eta}{\eta_f}\right) \]

\[ H_v = 1 - \frac{\eta}{M_g} \]

\[ H_s = \beta_o \beta_1 \exp(-\beta_o \xi) \]

\[ H_{dm} = \left(\frac{\sigma_{\max} J(s)}{\xi}\right)^{\gamma_{dm}} \]

and

\[ \eta = q / p' \]

\[ \eta_f = (1 + \frac{1}{c})M_f \]

\[ \xi = \int d\xi = \left|\frac{d\varepsilon^p}{d\tau}\right| d\tau \]

\[ \varsigma = p' \left(1 - \left(\frac{1}{1+c}\right)\frac{\eta}{M}\right)^{1/c} \]

\[ J(s) = e^{(\alpha(1-S_c))} \]

\[ \alpha, \beta_o, \text{ and } \beta_1 \text{ are material parameters and } \gamma_{dm} \text{ is a degradation parameter accounting for the fact that less plastic deformation occurs if a higher stress ratio has previously been reached. } \xi \text{ is the accumulated deviatoric plastic strain and } \frac{d\varepsilon^p}{d\tau} = \dot{\varepsilon}_r. \varsigma \text{ is the mobilized stress function. } J(s) \text{ provides an additional form of hardening due to partial saturation.} \]

For sand, plastic deformation during unloading is not a negligible effect. The higher the stress ratio for which unloading takes place, the bigger the amount of plastic deformation. The unloading plastic modulus is thus taken as

\[ H_v = H_w \left(\frac{\eta_u}{M_g}\right)^{\gamma_u} \]

where \(H_w\) and \(\gamma_u\) are material parameters and \(\eta_u\) is the stress ratio from which unloading takes place.

\[ H_b \text{ in equation (66) is defined as} \]

\[ H_b = w H_0 p' H_f H_{dm} \]

where \(w\) can be constant or it can be assumed as to be function of \(s\).

This modulus can be determined starting from a wetting path in which the material undergoes collapse (Tamagnini and Pastor, 2004), because the wetting path is an unloading stress path, hence the first term of (66) is zero, in accordance with the saturated case.
The last item required for specification of the model is the elastic matrix $D'$. In isotropic condition this matrix can be defined by two material properties, e.g., the bulk modulus $K$ and the shear modulus $G$.

It is obvious that as the mean confining stress $p$ increases, the granular material will tend to compact and stiffen. This behaviour can be incorporated by assuming that $K$ and $G$ are varying linearly with the mean confining stress, i.e.

$$K = K_{eso}p' \quad G = \frac{1}{3} K_{eso}p'$$

where both $K_{eso}$ and $K_{eso}$ are model parameters.

The dependence on suction is introduced also in the equations of yield and potential function. Experimental observations show that parameter $p_f$ is increasing with suction. Given the initial yield stress $p_{y0s}$ for saturated conditions, the dependence of $p_f$ on suction is assumed as:

$$p_f = p_{y0s} + RI \ s$$

The parameter $RI$ has to be determined by interpolation of experimental data to obtain an increasing function of suction when water saturation is less than one.

The applicability of this mathematical model requires the determination of ten material parameters by using experimental data. This has been performed for a laboratory test (Simoni and Schrefler (2001a, b)).

5. A Return-Mapping Algorithm

It is now shown how to integrate the above equations of the generalized plasticity model in a consistent, efficient and robust form. This possibility was first recognized by De Borst and Heeres (2000) for a strain driven plasticity model and then extended by Zhang et al. (2001) in the algorithm which is strain and suction driven. The task of such an algorithm within computational plasticity is to find the solution at time step $n+1$ with the known results at time step $n$, i.e.

$$\sigma_{n+1} = \sigma_n + \Delta \sigma_{n+1}^{trial} - \Delta \sigma_{n+1}^p$$

$$a_{n+1} = a(\sigma_{n+1}, \epsilon_{n+1}^p)$$

$$f(\sigma_{n+1}, a_{n+1}) = 0$$

where $f$ is the yield function which does not exist explicitly in the generalized plasticity
constitutive model, and

\[ \Delta \sigma_{n+1}^{trial} = D_{n+1}^e (\varepsilon_{n+1} - \varepsilon_n) \]
\[ \Delta \sigma_{n+1}^p = \Delta \lambda_{n+1} D_{n+1}^e \partial \ g_{n+1} \]  

with \( g \) the plastic potential, which also does not need to exist in generalized plasticity.

Discrete residual variables \( \mathbf{r} = [\mathbf{r}_a^T, \mathbf{r}_a^T, \mathbf{r}_f^T]^T \) are defined as

\[ r_a = \sigma_{n+1} - \sigma_n - \Delta \sigma_{n+1}^{trial} + \Delta \sigma_{n+1}^p \]  
\[ r_a = a_{n+1} - a(\sigma_{n+1}, \varepsilon_{n+1}) \]  
\[ r_f = f(\sigma_{n+1}, a_{n+1}) \]

and the stress update scheme then gives (\( k \) means iteration number)

\[ \partial_\varepsilon \mathbf{r}_{n+1}^k \Delta \mathbf{a}_{n+1}^k = -r_{n+1}^k, \quad a_{n+1}^k = a_{n+1}^0 + \Delta a_{n+1}^k \]

where

\[ a_{n+1}^0 = [\sigma_{n+1}, \sigma_{n+1}^T, a_{n+1}, \Delta \lambda_{n+1}]^T \]

and the consistent tangent is found to be

\[ d_\varepsilon \sigma = \partial_\varepsilon \sigma - \partial_\varepsilon \sigma [\partial_\varepsilon \mathbf{r}]^{-1} \partial_\varepsilon \mathbf{r} \]

As shown by De Borst and Heeres (2000) and Zhang et al. (2001), the algorithm cannot be easily adapted to generalized plasticity. Due to the fact that the yield function is not defined explicitly in the model, the residual equation (73c) cannot be used. Instead, the consistency equation is used, which for generalized plasticity is directly defined. In incremental form and for one mechanism (\( m \)), this equation reads

\[ \Delta \lambda_{n+1} = \left[ n_{n+1}^T \Delta \sigma \right]_{n+1} \]

With the definitions (19), the new discrete residual formulation can thus be expressed as

\[ r_a = \sigma_{n+1} - \sigma_n - \Delta \sigma_{n+1}^{trial} + \Delta \sigma_{n+1}^p \]  
\[ r_a = a_{n+1} - a(\sigma_{n+1}, \varepsilon_{n+1}, s_{n+1}) \]  
\[ r_b = \Delta \lambda_{n+1} \left( H(\sigma, \alpha, s)_{n+1} + n_{n+1}^T D_{n+1}^s n_{gL/U_{u+1}} - n_{n+1}^T D_{n+1}^s \Delta s_{n+1} \right) \]
\[ + \frac{1}{H_{b_{n+1}}} n_{n+1}^T D_{n+1}^s n_{gL/U_{u+1}} \Delta s_{n+1} \]

The stress update algorithm is essentially the same as in (75) where \( \mathbf{r} = [\mathbf{r}_a^T, \mathbf{r}_a^T, \mathbf{r}_f^T]^T \), while the consistent tangential stiffness matrix can still be obtained from equation (78).
6. Implementation of the Return-Mapping Algorithm for the Enhanced Constitutive Model

In this contribution a “Cutting Tangent Plane” algorithm (Simo and Hughes (1998)) has been adopted to bypass the rather cumbersome analytical derivation of gradients. It is noted that an alternative possibility would have been the numerical computation of the necessary gradients, see Pérez-Foguet et al. (1999).

The return-mapping algorithm with “General Closest Point Projection Iteration” is given as follows.

Step 0: Initialize:
\[ k = 0, \quad \varepsilon^{(0)}_{n+1} = \varepsilon^{(0)}_n, \quad \Delta \lambda^{(0)}_{n+1} = 0 \]  (81)

Step 1: Compute stresses and plastic strains:
\[ \Delta \sigma^{(0)}_{n+1} = D^{(0)}_{n+1} (\varepsilon^{(0)}_{n+1} - \varepsilon^{(0)}_n) = D^{(0)}_{n+1} \Delta \varepsilon_{n+1}, \]
\[ \sigma^{(0)}_{n+1} = \sigma^{(0)}_n + \Delta \sigma^{(0)}_{n+1}, \]
\[ \Delta \varepsilon^{(0)}_{n+1} = 0 \]  (82)

Step 2: Evaluate residuals:
\[ r^{(k)}_{\sigma_{n+1}} = \Delta \sigma^{(k)}_{n+1} - D^{(k)}_{n+1} (\Delta \varepsilon^{(k)}_{n+1} - \Delta \varepsilon^{(k)}_{n+1}) \]
\[ r^{(k)}_{\varepsilon_{n+1}} = -\Delta \varepsilon^{(k)}_{n+1} + \Delta \lambda^{(k)}_{n+1} D^{(k)}_{n+1} n_{gl/U_{s+1}}^{(k)} + \frac{\Delta s}{H_{\lambda^{(k)}_{n+1}}} n_{gl/U_{s+1}}^{(k)} \]
\[ r^{(k)}_{\lambda_{n+1}} = \Delta \lambda^{(k)}_{n+1} (H^{(k)}_{\lambda^{(k)}_{n+1}} + n^{(k)}_{n+1} D^{(k)}_{n+1} n_{gl/U_{s+1}}^{(k)}) \]
\[ -n^{(k)}_{n+1} D^{(k)}_{n+1} \Delta \varepsilon^{(k)}_{n+1} + \frac{\Delta s}{H_{\lambda^{(k)}_{n+1}}} n^{(k)}_{n+1} D^{(k)}_{n+1} n_{gl/U_{s+1}}^{(k)} \]  (83)

Step 3: Compute consistent tangent and standard matrices
\[ \Xi^{(k)}_{n+1} = \left[ D^{(k)}_{n+1}^{-1} + \Delta \lambda^{(k)}_{n+1} \partial_{\sigma} n_{gl/U_{s+1}}^{(k)} + \frac{\Delta s}{H_{\lambda^{(k)}_{n+1}}} \partial_{\sigma} n_{gl/U_{s+1}}^{(k)} \right]^{-1}, \]
\[ N^{(k)}_{n+1} = D^{(k)}_{n+1} \partial_{\sigma} n^{(k)}_{n+1}, \]
\[ M^{(k)}_{n+1} = n^{(k)}_{n+1} D^{(k)}_{n+1} \partial_{\sigma} n_{gl/U_{s+1}}^{(k)} + n_{gl/U_{s+1}}^{(k)} N^{(k)}_{n+1} \]  (84)

Step 4: Obtain increment to consistency parameter:
\[ d\Delta \lambda^{(k)}_{n+1} = \frac{R}{A} \]  (85)

where
\[ A = (H^{(k)}_{n+1} + n^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} + \]
\[ (\Delta \varepsilon^{(k)}_{n+1} N^{(k)}_{n+1} - \Delta \lambda^{(k)}_{n+1} M^{(k)}_{n+1}) \frac{\Delta s}{H^{(k)}_{b,n+1}} M^{(k)}_{n+1} ) \Xi^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} + \]
\[ R = (-\Delta \varepsilon^{(k)}_{n+1} N^{(k)}_{n+1} + \Delta \lambda^{(k)}_{n+1} M^{(k)}_{n+1} + \frac{\Delta s}{H^{(k)}_{b,n+1}} M^{(k)}_{n+1} ) \Xi^{(k)}_{n+1} r^{(k)}_{p,n+1} - r^{(k)}_{g,n+1} \]
\[ r^{(k)}_{p,n+1} = r^{(k)}_{p,n+1} + D^{(k)} n^{-(k)} n_{\varepsilon,n+1}(k) \]

Step 5: Compute incremental stresses and plastic strains:
\[ d\Delta \sigma^{(k)}_{n+1} = -\Xi^{(k)}_{n+1} (D^{(k)} n^{-(k)} n_{\varepsilon,n+1}^{(k)} + d\Delta \lambda^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} + r^{(k)}_{p,n+1}) \]
\[ d\Delta \varepsilon^{(k)}_{n+1} = -D^{(k)} n^{-(k)} (d\Delta \sigma^{(k)}_{n+1} + r^{(k)}_{\varepsilon,n+1}) \]

Step 6: Update state variables and consistency parameter:
\[ \Delta \sigma^{(k+1)}_{n+1} = \Delta \sigma^{(k)}_{n+1} + d\Delta \sigma^{(k)}_{n+1} \]
\[ \Delta \varepsilon^{(k+1)}_{n+1} = \Delta \varepsilon^{(k)}_{n+1} + d\Delta \varepsilon^{(k)}_{n+1} \]
\[ \Delta \lambda^{(k+1)}_{n+1} = \Delta \lambda^{(k)}_{n+1} + d\Delta \lambda^{(k)}_{n+1} \]

Step 7: Convergence determination:
If \[ \|d\Delta \sigma^{(k)}_{n+1}\| < Tol_1 \] and \[ \|d\Delta \varepsilon^{(k)}_{n+1}\| < Tol_2 \] then
Stop
else \[ k \leftarrow k+1 \] and goto Step 2.

A disadvantage of using the Cutting Tangent Plane algorithm is that it cannot be linearised consistently, but as a good approximation we can use
\[ d\sigma^{(k)}_{n+1} = \left( \Xi^{(k)}_{n+1} + \frac{\Xi^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} n^{(k)}_{n+1}}{H^{(k)}_{n+1} + n^{(k)}_{n+1} \Xi^{(k)}_{n+1} n_{gl/U_{n+1}}} \right) d\varepsilon - \frac{\Xi^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} H^{(k)}_{n+1} ds}{H^{(k)}_{b,n+1} (H^{(k)}_{n+1} + n^{(k)}_{n+1} \Xi^{(k)}_{n+1} n_{gl/U_{n+1}})} \]

where
\[ n^{(k)}_{n+1} = n^{(k)}_{n+1} (n^{(k)}_{n+1} = n^{(k)}_{n+1} + \partial \sigma n^{(k)}_{n+1}) \]

Thus the relationship of stress and strain can be expressed
\[ \frac{d\sigma}{d\varepsilon}_{n+1} = \Xi^{(k)}_{n+1} - \frac{\Xi^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} n^{(k)}_{n+1} \Xi^{(k)}_{n+1}}{H^{(k)}_{n+1} + n^{(k)}_{n+1} \Xi^{(k)}_{n+1} n_{gl/U_{n+1}}} = D^{(k)}_{n+1} \]

and the relationship of stress and suction is obtained at the same time
\[ \frac{d\sigma}{ds}_{n+1} = -\frac{\Xi^{(k)}_{n+1} n_{gl/U_{n+1}}^{(k)} H^{(k)}_{n+1}}{H^{(k)}_{b,n+1} (H^{(k)}_{n+1} + n^{(k)}_{n+1} \Xi^{(k)}_{n+1} n_{gl/U_{n+1}})} = -D^{(k)}_{n+1} \]

where \[ \Xi^{(k)}_{n+1} \] is defined in equation (84).
The explanation of $D_{s+1}^{(ep)}$ in equation (92) can be given by one dimensional analysis. In this case, $D_{s+1}^{(ep)}$ decreases to

$$D_{s}^{(ep)} = \frac{1}{H_b} \frac{HE}{(H + E)} = \frac{E^T}{H_b}, \quad \text{with} \quad E^T = \frac{HE}{(H + E)}$$

(93)

where $E$ and $E^T$ are the elastic modulus and tangential moduli.

The examples discussed at the end of this contribution show that a nearly quadratic convergence can be obtained. Because of its explicit nature the Cutting Tangent Plane algorithm is not unconditionally stable. However, no numerical instabilities were encountered in the computation of the example problems.

Up to now, the incremental plasticity relations are defined in terms of generalized stress invariants, i.e. in a physically meaningful stress subspace with its related strain quantities. However, in computations it is necessary to transform the relation to a Cartesian co-ordinate system. Further, relevant incremental plastic strain quantities in the Cartesian system have to be transformed to the original $p'qs'$-space for evaluating the work hardening or strain hardening parameters, in particular for the prediction of the critical hardening modulus when localisation takes place. Also, since the subspace has different dimensions from that in the Cartesian system, the reversed transformation can not be done simply by inverting the forward transformation.

The explicit expressions for the definitions in above can be given as

$$\partial_{s} x = W \cdot V$$

(94)

$$\partial_{s\sigma} x = W \cdot R \cdot W^T + c_1 R(p') + c_2 R(q) + c_3 R(s)$$

(95)

for any scalar-valued function $y$, and the matrices have the following forms

$$W = \begin{bmatrix}
\partial_{\sigma_1} p' & \partial_{\sigma_1} q & \partial_{\sigma_1} s \\
\partial_{\sigma_2} p' & \partial_{\sigma_2} q & \partial_{\sigma_2} s \\
\vdots & \vdots & \vdots \\
\partial_{\sigma_n} p' & \partial_{\sigma_n} q & \partial_{\sigma_n} s
\end{bmatrix},$$

$$R = \begin{bmatrix}
\partial_{p'p'} x & \partial_{p'q} x & \partial_{p's} x \\
\partial_{qq} x & \partial_{qq} x & \partial_{ss} x \\
\partial_{sp} x & \partial_{qp} x & \partial_{ss} x
\end{bmatrix},$$

(96)

$$V = \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix},$$

(97)

$v_1 = \partial_{p'} x, \quad v_2 = \partial_{q} x, \quad v_3 = \partial_{s} x$

while the operator matrix $R$ is defined as
\[ \mathbf{K} = \begin{bmatrix} \partial_{\sigma_1\sigma_{11}} & \partial_{\sigma_1\sigma_{12}} & \cdots & \partial_{\sigma_1\sigma_{13}} \\ \partial_{\sigma_2\sigma_{11}} & \partial_{\sigma_2\sigma_{12}} & \cdots & \partial_{\sigma_2\sigma_{13}} \\ \cdots & \cdots & \cdots & \cdots \\ \partial_{\sigma_{11}\sigma_{11}} & \partial_{\sigma_{11}\sigma_{12}} & \cdots & \partial_{\sigma_{11}\sigma_{13}} \end{bmatrix} \]  

(98)

7. Numerical Tests

7.1 Comparison with experimental data

The data used have been obtained from the extensive experimental investigation conducted by Geiser (1999). The soil used by Geiser is a slime of the Sion region in Switzerland. It is composed by slime (72%), clay (8%) and sand (20%). First the saturated behaviour.

The list of unsaturated deviatoric normal consolidated specimens of Sion slime is reported in Table 1.

Figures 1(8) and 2(9) show the behaviour of the unsaturated soil analysed in drained conditions with the same value of suction 100KPa and different cell pressures \( \sigma_3 \). The behaviour of unsaturated specimens is similar to that analysed in saturated conditions, but the suction products a type of behaviour weakly overconsolidated. There is a little peak and then a decrease of the behaviour stress \( q \) until a last lower value. It is clear for the specimens at suction of 100Kpa 9 (see Figure 8).
Table 1. The list of unsaturated deviatoric normal consolidated specimens of Sion slime

<table>
<thead>
<tr>
<th>Cell pressure σ3 (KPa)</th>
<th>s</th>
<th>Δq / Δp'</th>
<th>e_i</th>
<th>e_f</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>100</td>
<td>3</td>
<td>0.802</td>
<td>0.66</td>
</tr>
<tr>
<td>500</td>
<td>100</td>
<td>3</td>
<td>0.712</td>
<td>0.63</td>
</tr>
<tr>
<td>600</td>
<td>50</td>
<td>3</td>
<td>0.755</td>
<td>0.62</td>
</tr>
<tr>
<td>600</td>
<td>100</td>
<td>3</td>
<td>0.788</td>
<td>0.64</td>
</tr>
<tr>
<td>600</td>
<td>280</td>
<td>3</td>
<td>0.712</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Figure 1 Deviatoric stress versus axial strain in triaxial test normal consolidation partially saturated specimen in drained conditions with suction 100KPa
7.2 Application to an oedometric test

The numerical algorithm developed and the material model previously summarized are now used for the solution of a laboratory experiment performed at IKU Petroleum Research, Trondheim (Papamichos and Schei (1998), Papamichos et al. (1998)) on behalf of AGIP (Italian National Petroleum Company).

The problem deals with a silty consolidated sandstone sample extracted from a gas bearing formation in the Northern Adriatic basin at a depth of 3400 m. Effective porosity, in situ water saturation and irreducible saturation of the material were independently obtained. Subsequently the specimen underwent an oedometric test.

The loading process was scheduled as follows: the sample at in situ saturation (0.38-0.45) is firstly stressed with an initial hydrostatic phase presenting $\sigma_r$-rate equal to 0.01 MPa/s until $\sigma_r=0.5$ MPa. This is followed by a uniaxial phase with $\sigma_z$-rate of 0.004 MPa/s until $\sigma_z$ reaches 35 MPa; the sample is then held at constant stress level and water is injected for 25 hours until full saturation is attained. This procedure is simulated through the specification of change of saturation (suction) from 0.38 to 1.0. During this period of time, volumetric changes of the specimen are recorded, as during the phases of stress changes. Once full saturation is reached, a second uniaxial phase, at constant water content, with stress rate of 0.004 MPa/s till about 110 MPa is performed. The test includes also unloading cycles to determine the elastic behaviour and recoverable deformation. The water injection test (hydric-path) simulates the behaviour of
the gas reservoir rock during artificial water injection or during the flooding associated with gas extraction. For this reason the axial stress level at which the sample is injected is representative of the vertical stress in reservoir conditions. In the absence of other information, we assume that gas pressure during the test maintains the same value (reference or zero pressure).

The parameters used for the generalized model are shown in Table 2.
Table 2. Material parameters of the second example

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk modulus of water $K_w$ (Pa)</td>
<td>2000E6</td>
</tr>
<tr>
<td>Bulk modulus of solid grain $K_s$ (Pa)</td>
<td>5.0E10</td>
</tr>
<tr>
<td>Bulk modulus of gas $K_g$ (Pa)</td>
<td>1.0E6</td>
</tr>
<tr>
<td>Solid grain density $\rho_s$ (Kg/m$^3$)</td>
<td>2000</td>
</tr>
<tr>
<td>Mass density of water $\rho_w$ (Kg/m$^3$)</td>
<td>1000</td>
</tr>
<tr>
<td>Mass density of gas $\rho_g$ (Kg/m$^3$)</td>
<td>15.5</td>
</tr>
<tr>
<td>Initial porosity $n$</td>
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<tr>
<td>Dynamic permeability of water $k/\mu_w$ (m$^2$/Pa/s)</td>
<td>2.548E-11</td>
</tr>
<tr>
<td>Dynamic permeability of gas $k/\mu_g$ (m$^2$/Pa/s)</td>
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</tr>
<tr>
<td>Gravitational acceleration $g$ (m/s$^2$)</td>
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</tr>
<tr>
<td>Irreducible saturation used in $S_w$-$P_c$ relation $S_{irr}$</td>
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<tr>
<td>Coefficient $\lambda$ used in $S_w$-$P_c$ relationship</td>
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</tr>
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<td>Bubbling pressure used in $S_w$-$P_c$ relationship $p_b$ (Pa)</td>
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<td>Reference pressure used in $S_w$-$P_c$ relationship $p_{ref}$ (Pa)</td>
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<td>Parameter $c$ in enhanced Model</td>
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<td>Parameter $K_{evo}$ in enhanced Model</td>
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<td>Parameter $K_{eso}$ in enhanced Model</td>
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<td>Parameter $H_{d0}$ in enhanced Model</td>
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</tr>
<tr>
<td>Parameter $M_g$ in enhanced Model</td>
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<tr>
<td>Parameter $M_f$ in enhanced Model</td>
<td>1.3</td>
</tr>
<tr>
<td>Parameter $\gamma_{DM}$ in enhanced Model</td>
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</tr>
<tr>
<td>Parameter $\gamma_u$ in enhanced Model</td>
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</tr>
</tbody>
</table>

Figure 3 shows axial stress vs. volumetric strain as results from the experiment. For identification purposes, the experimental curves are slightly changed by eliminating the unloading/reloading cycles. Also the experimental response has been slightly idealised by assuming 30 points, including of the initial known conditions, as shown in Figure 4 together with the model response. The agreement is satisfactory, in particular for the increase of volumetric strain caused by water injection.
Figure 3. Oedometric test with water injection: axial stress vs. volumetric strain. (redrawn from Papamichos and Schei, 1998)

Figure 4 Comparison between numerical results and lab response
7.3 The complete mathematical model for reservoir analysis

A gas reservoir of the type investigated in this section contains gaseous and liquid phases at the same time and within it pressure gradients are possible. As a consequence, the pressure measurements usually performed can not represent what happens in the whole reservoir. Moreover the reservoirs are in hydraulic continuity with confining aquifers and aquitards. A complete mathematical model to simulate the mechanical behaviour of such a multiphase system needs therefore a mass balance equation for all present fluids together with momentum balance equations for fluids and the mixture. All these equations are coupled owing to the interactions between the fluids and the solid (see Lewis and Schrefler (1998)).

We now perform a numerical simulation of a well-documented model reservoir, which represents an interesting validation tests for the presented numerical procedure.

The reservoir is a hypothetical one (Figure 5), which has already been studied in (Evangelisti and Poggi (1970)) and, with an additional clay layer, in (Lewis and Schrefler (1998)).

The problem has the following boundary conditions: R=0m, axial symmetric boundary conditions; R=14005m, constrained boundary conditions; Z=0m, constrained boundary conditions; Z=1080m, zero water and gas pressure conditions; Z=0m, undrained boundary conditions for both water and gas; R=14005m, constant water and gas pressure conditions.

The gas production history is that of a real reservoir and is depicted in Figure 6.

The integration process is divided into 300 time steps in the computation. To study the advantages of the new algorithm developed here, both the traditional \( N-R \) algorithm and the new one are used during the numerical test. From the numerical experiments, it appears that during the period of water injection and soon after it, the behaviour is strongly nonlinear. With the traditional method, the total iteration number for the full calculation is 13567, while for the new one it is 6981. The iteration numbers for different tolerance and typical convergent process are shown in Figures (9a, b). There we have not only the solid skeleton constitutive behaviour, but also the nonlinear properties of the water and gas flow. It can be observed that the new algorithm has a advantage in the convergence speed (computing cost saves about 50%).

The parameters used in the constitutive model are illustrated in Table 3.

The vertical displacement results along the horizontal direction on the ground surface with different time are shown in Figure 8. The results of vertical displacements at the different places of the ground surface versus time can be seen in Figure 9.

The water and gas pressure versus time spending in the reservoir is presented in Figure 10. Figure 11 shows the reservoir saturation history which lasts for 50 years. It can be
observed that at depletion of the wells the flow fields still present gradients which require
pressure changes, fluxes of fluids and solid deformation developing for some time after
the end of gas extraction.

Figure 5  Geometry of the reservoir

Figure 6  Gas production history for numerical simulation
Figure 7 Comparison of computational cost of the second example
Table 3. Material parameters of the third example

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Material I</th>
<th>Material II</th>
<th>Material III</th>
<th>Material IV</th>
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Figure 8 Vertical displacements along the ground surface

Figure 9 Vertical subsidence at different places of the ground
8. Conclusions

This paper shows that the application of an implicit stress update algorithm together with a tangent operator consistent with it can be successfully applied to an enhanced generalized plasticity model which is strain and suction driven for partially saturated soils. The model used is an enhanced Pastor-Zienkiewicz model to take into account of the hydraulic constitutive relationship, hydraulic hysteresis, and a new term of plastic strain.
to present the mechanical behaviour of partially saturated sands. Numerical examples show also the good convergence behaviour of the algorithm.

**Acknowledgements**

The first author acknowledges the supports of the National Natural Science Foundation (10225212, 50178016) and the State Educational Committee of China.

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