Double diffusion in rotating porous media under general boundary conditions

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Abstract

In this article we study a binary fluid saturating a rotating porous medium; the fluid is modeled according to Darcy-Brinkman law and the boundary conditions are rigid or stress-free on the velocity field and of Robin type on temperature and solute concentration.

We determine the threshold of linear instability and its dependence on Taylor and Darcy numbers. Using a Lyapunov function we prove analytically, under certain assumptions, the coincidence of linear and nonlinear thresholds. A second Lyapunov function allows us to prove numerically the coincidence of the two thresholds with weaker assumptions on the parameters.

We show that in the particular limit case of fixed heat and solute fluxes this system has a remarkable feature: the wave number of critical cells goes to zero when the Taylor number is below a threshold. Above such threshold, the wave number is non-zero when the Darcy number belongs to a finite interval. These phenomena could perhaps be tested experimentally.

Keywords: Porous media, Binary fluids, Stability, Rotation, Robin, Energy stability.

1 Introduction

The problem of thermal convection of a binary fluid saturating a porous medium is a subject of intense research [34, 40, 44]. Double-diffusive flow in porous media has immediate applications in geophysical problems such as contaminant transport in groundwater [15] and exploitation of geothermal reservoirs [20, 36], but it also occurs in astrophysics [27, 17], metallurgy [13], and electrochemistry [12]. In many of these problems, rotation of the medium plays an important role (see e.g. [21, 22, 23, 24, 25, 26]).

Other than a linear analysis of the problem, nonlinear analysis by means of Lyapunov functions can give results of global stability, and in any case gives indications on the radius of the basin of attraction of the equilibrium. The technique of Lyapunov functions has been used to investigate the coincidence between linear instability and non-linear stability threshold for a fluid modeled by Darcy's law in the case of a rotating porous layer [40, 38]. Similar problems have been extensively studied in fluid dynamics [30, 29, 16].

Many phenomena in porous media cannot be described by the simple Darcy law and require the inclusion of Brinkman's term, which gives equations better suited to describe the fluid motion in a sparsely packed media [45]. The case of a rotating binary fluid modeled by Darcy-Brinkman law has been treated by [18] under the hypothesis of stress free boundary conditions on velocity and fixed temperature and solute concentration. Even though such boundary conditions allow an analytic treatment of the problem, they still are too restrictive for physical models.

It is known that in the case of Neumann boundary conditions on the temperature the critical wave number of the perturbation goes to zero (see [35] for the simple Bénard system and [32, 33] for the case

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of binary fluids and porous media). This physically means that the convection will take place in a cell with the largest extension allowed by the system or the experimental setup. In [10] the writers have found an interesting analytical and physical result for a rotating porous layer: under Neumann boundary conditions, the wave number of the critical perturbation goes to zero only when the rotation number is sufficiently low. The same result holds in the rotating Bénard problem [8]. This phenomenon is probably suitable for experimental verification. The novelty of this work with respect to the literature cited above (in particular [10]) is that here the fluid is modeled by the Brinkman-Darcy law; moreover we consider the effect of the diffusion of a solute. The presence of the solute field makes the energy analysis of the system much more complex.

In this article we investigate the problem of a rotating binary fluid in a porous medium with Robin conditions on temperature and solute concentration. Such boundary condition correspond to the physically relevant cases in which the media surrounding the porous layer have a relatively small conductivity and heat capacity. For the same reason the solute flow at the boundaries is better modeled by a Robin-type law [14]. The relevance of this type of boundary conditions is stated in recent works (see e.g. [1]).

In section 2 we describe the problem and obtain the differential equations and boundary conditions. We investigate in section 3 the associated linear system and the validity of the principle of exchange of stability (PES) i.e. we verify when instability can arise only as stationary convection. In particular we prove analytically that a strong PES (SPES) — that is the spectrum of the linear system is always real — holds when $\epsilon\mathrm{Le} = 1$ and also when the fluid is homogeneous. We find numerical evidence that PES always holds when $\epsilon\mathrm{Le} < 1$ while, when $\epsilon\mathrm{Le} > 1$, overstability appears for some choice of the parameters. As far as we know determining exact condition for PES or SPES is still an open problem.

Under the hypothesis of PES, we derive and numerically plot critical stability parameters for a variety of boundary conditions. In particular we obtain that, in the case of fixed heat fluxes, the critical wave number is non-zero when the Darcy number belongs to a finite interval depending on the rotation speed. In section 4 we use two Lyapunov functions to investigate the non-linear stability of the stationary solution when temperature and solute concentration are subject to the same Robin boundary conditions. Using the first Lyapunov function we prove analytical coincidence of linear and non-linear threshold when $\epsilon\mathrm{Le} = 1$; using the second Lyapunov function we provide numerical evidence of coincidence in all other cases. In both cases global stability is proved for some values of the physical parameters.

2 The problem

We consider, in a reference frame $Oxyz$, with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, a fluid saturating a horizontal layer of a porous medium bounded by the surfaces $z = \pm d/2$. The layer rotates around the z-axis with angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$ and the system is subject to gravity $\mathbf{g} = -g\mathbf{k}$.

We assume a linear dependency of the fluid density ρ_f on its temperature T and on the concentration C of a solute, $\rho_f = \rho_0[1 - c(T - T_0) + c'(C - C_0)]$. Here c, c' are positive coefficients, ρ_0, T_0, C_0 are a reference density, temperature, and solute concentration respectively.

If the fluid flow is governed by Brinkman's law, then in the Oberbeck-Boussinesq approximation the equations describing the system are

$$\begin{cases} \nabla P = -\rho_f g\,\mathbf{k} - \frac{\mu_1}{K}\mathbf{v} + \mu_2\Delta\mathbf{v} - 2\frac{\rho_0}{\varepsilon}\boldsymbol{\omega}\times\mathbf{v} \\ \nabla\cdot\mathbf{v} = 0 \\ \frac{1}{M}\,\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = k\,\Delta T \\ \varepsilon\,\frac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = k'\,\Delta C. \end{cases}$$

Here $P = p_1 - \frac{1}{2}\rho_0[\boldsymbol{\omega}\times\mathbf{x}]^2$, p_1 is the pressure, $\mathbf{x} = (x, y, z)$ and $\mathbf{v} = (U, V, W)$ denotes the seepage velocity of the fluid. The last two terms of the first equation are the Brinkman term and the Coriolis acceleration.

The other quantities appearing in the system are: porosity (ε), viscosity and an effective viscosity (μ_1, μ_2), permeability of the medium (K), salt diffusivity (k'). Quantities k and M are an effective thermal conductivity and the ratio of heat capacities; they depend on the different thermal properties of the fluid and the porous medium according to

$$k = \frac{(1-\varepsilon)k_s + \varepsilon k_f}{(\rho\, c_p)_f}, \qquad M = \frac{(\rho\, c_p)_f}{(1-\varepsilon)(\rho\, c)_s + \varepsilon(\rho\, c_p)_f},$$

where c_p is the specific heat of the fluid at constant pressure and subscripts f, s denote fluid and solid (porous matrix) values respectively. More details about these equations and physical quantities can be found, e.g., in [34] or [40].

The velocity field is subject to

$$\begin{array}{ll} U = V = W = 0, & \text{at rigid boundaries and} \\ U_z = V_z = W = 0, & \text{at stress free boundaries,} \end{array}$$

where subscript z denotes the partial derivative along the z-axis.

We study a basic motionless state in which temperature and concentration at the boundaries are given by

$$\begin{cases} T(x,y,-d/2) = T_1, & T(x,y,d/2) = T_2 \\ C(x,y,-d/2) = C_1, & C(x,y,d/2) = C_2 \end{cases}$$

with $T_0 = (T_1 + T_2)/2$ and $C_0 = (C_1 + C_2)/2$. We also assume that $T_1 > T_2$ and $C_1 > C_2$, that is the fluid is *heated and salted from below.* Observe that the values of T_1, T_2 do not necessarily correspond to an hypothesis of fixed values at the boundaries (i.e. isothermal surfaces), but are consequence of the more general and physically meaningful boundary conditions

$$\begin{array}{ll} \alpha_-(T_z + \beta)d - (1-\alpha_-)(T - T_1) = 0, & \text{on } z = -d/2 \\ \alpha_+(T_z + \beta)d + (1-\alpha_+)(T - T_2) = 0, & \text{on } z = d/2, \end{array} \tag{1}$$

with $\alpha_+, \alpha_- \in [0,1]$ and $\beta = (T_1 - T_2)/d$. This boundary conditions are such that by varying the parameter α_+ (or α_-) we get conditions of fixed temperature, fixed heat flux, or Newton-Robin (finite conductivity) for, respectively, $\alpha_+ = 0$, $\alpha_+ = 1$, $\alpha_+ \in (0,1)$ at the boundary $z = d/2$ (or $z = -d/2$).

Similarly, we do not make the hypothesis of isosolutal surfaces, but we assume the boundary conditions

$$\begin{array}{ll} \alpha'_-(C_z + \beta')d - (1-\alpha'_-)(C - C_1) = 0, & \text{on } z = -d/2 \\ \alpha'_+(C_z + \beta')d + (1-\alpha'_+)(C - C_2) = 0, & \text{on } z = d/2. \end{array} \tag{2}$$

with $\alpha'_+, \alpha'_- \in [0,1]$, and $\beta' = (C_1 - C_2)/d$.

The above boundary conditions (1) and (2) are such that, for any choice of $\alpha_\pm, \alpha'_\pm$, the basic motionless state is given by

$$m_0 = \{\mathbf{v} = 0, \quad T = T_0 - \beta z, \quad C = C_0 - \beta' z, \quad \nabla P = \rho_0(1 + (c\,\beta - c'\,\beta')\,z\,\mathbf{g})\}.$$

With a suitable choice of units, the equations describing the evolution of a disturbance to m_0 are

$$\begin{cases} \nabla p = (\mathrm{R}\,\theta - \mathrm{C}\,\gamma)\,\mathbf{k} - \mathbf{u} + \mathrm{Da}\,\Delta\mathbf{u} - \mathrm{T}\,\mathbf{k}\times\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \theta_t + \mathbf{u}\cdot\nabla\theta = \mathrm{R}\,w + \Delta\theta \\ \epsilon\,\gamma_t + \mathbf{u}\cdot\nabla\gamma = \mathrm{C}\,w + \frac{1}{\mathrm{Le}}\Delta\gamma, \end{cases} \tag{3}$$

where $p, \mathbf{u} = (u,v,w), \theta, \gamma$ are respectively the perturbations to pressure, velocity, temperature, and concentration, and the following dimensionless quantities appear

$$\mathrm{T} = 2\,\frac{K\,\omega}{\varepsilon\,\mu_1}, \quad \mathrm{R}^2 = \frac{c\,\beta\,g\,d^2\,K}{\mu_1\,k}, \quad \mathrm{C}^2 = \frac{c'\,\beta'\,g\,d^2\,K}{\mu_1\,k},$$

where T^2 is the Taylor-Darcy number, $\mathrm{R}^2, \mathrm{C}^2$ are the thermal and solutal Rayleigh numbers, proportional to temperature and solute gradient in the basic state m_0 respectively. Moreover,

$$\mathrm{Da} = \frac{\mu_2\, K}{\mu_1\, d^2}, \quad \epsilon = \varepsilon\, M, \quad \mathrm{Le} = \frac{k}{k'},$$

are the Darcy number, normalized porosity, and Lewis number. We eliminate the pressure field, taking the third component of the curl and double curl of the first equation, and use the substitution $\gamma \to \mathrm{Le}\,\gamma$, obtaining finally

$$\begin{cases} 0 = -\zeta + \mathrm{Da}\,\Delta\zeta + \mathrm{T}\,w_z \\ 0 = \mathrm{R}\,\Delta^*\theta - \mathrm{LeC}\,\Delta^*\gamma - \Delta w + \mathrm{Da}\,\Delta\Delta w - \mathrm{T}\,\zeta_z \\ \theta_t + \mathbf{u}\cdot\nabla\theta = \mathrm{R}\,w + \Delta\theta \\ \epsilon\mathrm{Le}\,\gamma_t + \mathrm{Le}\,\mathbf{u}\cdot\nabla\gamma = \mathrm{C}\,w + \Delta\gamma, \end{cases} \tag{4}$$

where $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{u}$ is the third component of the vorticity, and $\Delta^* = \partial^2/\partial^2 x + \partial^2/\partial^2 y$. The corresponding boundary condition for the perturbation fields are

$$\begin{array}{ll} w = w_{zz} = \zeta_z = 0 & \text{on stress-free boundaries} \\ w = w_z = \zeta = 0 & \text{on rigid boundaries} \\ \alpha_\pm\theta_z \pm (1-\alpha_\pm)\theta = 0, & \text{on } z = \pm 1/2 \\ \alpha'_\pm\gamma_z \pm (1-\alpha'_\pm)\gamma = 0 & \text{on } z = \pm 1/2. \end{array} \tag{5}$$

We perform a linear instability analysis of (4) in section 3, and two energy analyses of the system, corresponding to two different choices of the Lyapunov function (obtained using two field transformations) in sections 4.1 and 4.2.

3 Linear instability

A linear instability analysis of system (4) was performed in [18] for stress-free, isothermal, isosolutal boundary condition. With such choice of boundary conditions, the solutions are analytical. In this work we investigate boundary conditions on temperature and solute concentration of Robin type, and also combinations of rigid and stress-free boundaries. In these cases solutions can only be found numerically. We perform our numerical investigation using the Chebyshev-tau method described in [5].

3.1 Associated linear system

As customary, we assume that the perturbation fields are periodic in the x, y directions with periodicity cell $\Omega = [0, 2\pi/a_1] \times [0, 2\pi/a_2] \times [-1/2, 1/2]$, that is we assume that solutions are of the kind

$$f(x,y,z,t) = e^{\sigma t} F(z) g(x,y), \quad \text{with } \Delta^\star g + a^2 g = 0,$$

where f stands for the fields ζ, w, θ, γ, F stands for, respectively, Z, W, Θ, Γ, and $a^2 = a_1^2 + a_2^2$ is the wave number of the perturbation. Under these hypothesis, the linear equations associated to system (4) are

$$\begin{cases} 0 = -Z + \mathrm{Da}\,(D^2 - a^2)\,Z + \mathrm{T}\,DW \\ 0 = -\mathrm{R}\,a^2\,\Theta + \mathrm{LeC}\,a^2\,\Gamma - (D^2 - a^2)\,W + \mathrm{Da}\,(D^2 - a^2)^2 W - \mathrm{T}\,DZ \\ \sigma\Theta = \mathrm{R}\,W + (D^2 - a^2)\,\Theta \\ \sigma\,\epsilon\mathrm{Le}\,\Gamma = \mathrm{C}\,W + (D^2 - a^2)\,\Gamma, \end{cases} \tag{6}$$

where D denotes the derivative with respect to z. Boundary conditions for the above fields are

$$\begin{array}{ll} W = D^2W = DZ = 0 & \text{on stress-free boundaries} \\ W = DW = Z = 0 & \text{on rigid boundaries} \\ \alpha_\pm D\Theta \pm (1-\alpha_\pm)\Theta = 0 & \text{on } z = \pm 1/2 \\ \alpha'_\pm D\Gamma \pm (1-\alpha'_\pm)\Gamma = 0 & \text{on } z = \pm 1/2. \end{array} \tag{7}$$

3.2 Overstability

When the fluid is homogeneous, system (6) becomes

$$\begin{cases} 0 = -Z + \mathrm{Da}\,(D^2 - a^2)\,Z + \mathrm{T}\,DW \\ 0 = -\mathrm{R}\,a^2\,\Theta - (D^2 - a^2)\,W + \mathrm{Da}\,(D^2 - a^2)^2 W - \mathrm{T}\,DZ \\ \sigma\Theta = \mathrm{R}\,W + (D^2 - a^2)\,\Theta. \end{cases} \tag{8}$$

By taking the scalar product of the first two equations with Z and W respectively, keeping in mind boundary conditions (7) on W and Z, one shows that

$$\mathrm{Da}\,\|DZ\|^2 + (1 + \mathrm{Da}\,a^2)\|Z\|^2 = \mathrm{T}(DW, Z) \in \mathbb{R}$$

$$\mathrm{Da}\,\|D^2W\|^2 + (1 + 2\,\mathrm{Da}\,a^2)\|DW\|^2 + \\ + a^2(1 + \mathrm{Da}\,a^2)\|W\|^2 + \mathrm{T}\,(Z, DW) = \mathrm{R}\,a^2\,(\Theta, W) \in \mathbb{R}$$

where (A, B) and $\|A\|$ denote respectively the usual scalar product and norm in $L^2(-1/2, 1/2)$. From the last equation of (8), using boundary conditions (7) on Θ, one has

$$\sigma\|\Theta\|^2 = \mathrm{R}(W, \Theta) - \|D\Theta\|^2 - a^2\|\Theta\|^2 - S(\Theta, \alpha_+, \alpha_-) \in \mathbb{R},$$

where S is a real and non negative boundary term given, for a generic field A, by

$$S(A, \alpha_+, \alpha_-) = \begin{cases} \frac{1-\alpha_+}{\alpha_+} A(\frac{1}{2})^2 + \frac{1-\alpha_-}{\alpha_-} A(-\frac{1}{2})^2, & \text{for } \alpha_+ > 0, \alpha_- > 0 \\ \frac{1-\alpha_-}{\alpha_-} A(-\frac{1}{2})^2, & \text{for } \alpha_+ = 0, \alpha_- > 0 \\ \frac{1-\alpha_+}{\alpha_+} A(\frac{1}{2})^2, & \text{for } \alpha_+ > 0, \alpha_- = 0 \\ 0, & \text{for } \alpha_+ = \alpha_- = 0. \end{cases}$$

Since Θ cannot vanish (unless also W and Z vanish), it follows that $\sigma \in \mathbb{R}$, that is, when the fluid is homogeneous the strong PES holds.

Considering the non homogeneous case described by (6), using similar arguments we obtain $(DW, Z) \in \mathbb{R}$ and $a^2(\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma, W) \in \mathbb{R}$. Adding equation $(6)_3$ times R with $(6)_4$ times $-$LeC we obtain

$$\sigma\Big(\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma + (1 - \epsilon\,\mathrm{Le})\mathrm{LeC}\,\Gamma\Big) = (\mathrm{R}^2 - \mathrm{LeC}^2)W + (D^2 - a^2)(\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma).$$

Taking the scalar product of this equation with $\mathrm{R}\Theta - \mathrm{LeC}\Gamma$, and under the assumption $\epsilon\,\mathrm{Le} = 1$, we obtain

$$(\sigma + a^2)\|\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma\|^2 = (\mathrm{R}^2 - \mathrm{LeC}^2)(W, \mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma) + \\ + (D^2(\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma), \mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma). \tag{9}$$

The last term equals

$$-\|D(\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma)\|^2 - \mathrm{R}^2 S(\Theta, \alpha_+, \alpha_-) - \mathrm{Le}^2\mathrm{C}^2 S(\Gamma, \alpha'_+, \alpha'_-) + \\ + \mathrm{R}\,\mathrm{LeC}\left[\tfrac{1-\alpha_+}{\alpha_+}\Theta_+\overline{\Gamma}_+ + \tfrac{1-\alpha_-}{\alpha_-}\Theta_-\overline{\Gamma}_- + \tfrac{1-\alpha'_+}{\alpha'_+}\overline{\Theta}_+\Gamma_+ + \tfrac{1-\alpha'_-}{\alpha'_-}\overline{\Theta}_-\Gamma_-\right],$$

where $\bar{z}$ denotes the complex conjugate of z and the subscripts $\pm$ indicate that the functions are evaluated in $\pm 1/2$. The expression between square brackets is guaranteed to be real only when $\alpha_+ = \alpha'_+$ and $\alpha_- = \alpha'_-$, that is when the temperature and solute fields are subject to the same boundary conditions. Under these hypotheses, the imaginary part of (9) becomes

$$\Im(\sigma)\|\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma\|^2 = 0.$$

Since it can be easily shown that $\|\mathrm{R}\,\Theta - \mathrm{LeC}\,\Gamma\| = 0$ implies also $W = Z = 0$ and in turn $\Theta = \Gamma = 0$, the above equality implies $\Im(\sigma) = 0$. This proves that in the binary fluid case, the SPES holds when $\epsilon\mathrm{Le} = 1$ and temperature and solute are subject to the same Robin boundary conditions.

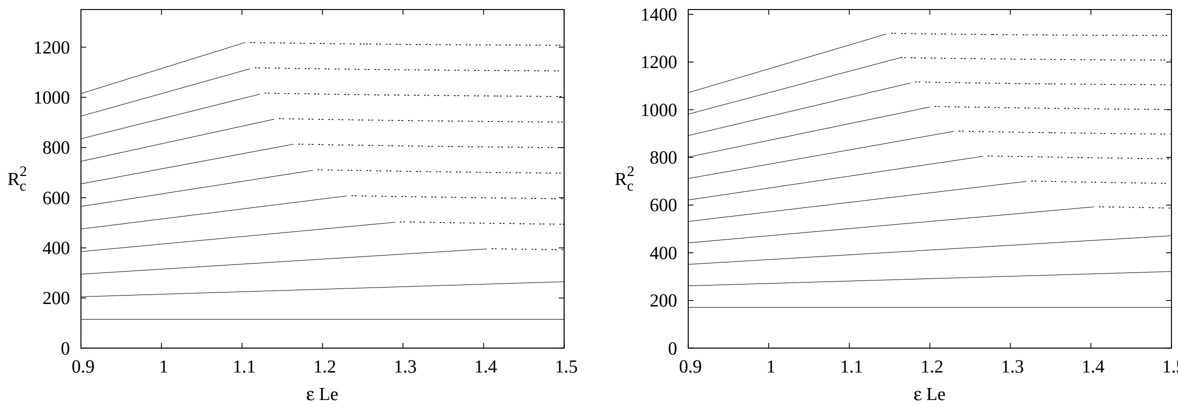

Figure 1: Critical Rayleigh number as functions of ϵLe. Dotted lines correspond to the onset of overstability. The plots are for stress free (left) and rigid (right) boundary conditions, with $\alpha = \alpha' = \frac{1}{2}$, $\mathrm{T}^2 = 10$, $\mathrm{Da} = 0.1$, $\epsilon = 1$. The curves are computed for $\mathrm{C}^2 = 0, 100, ..., 1000$ (from bottom to top).

We find also a strong numerical evidence supporting the validity of the SPES for $\epsilon\mathrm{Le} < 1$, even if we cannot prove it analytically. We note that this behaviour is coherent with the results shown in [31], where SPES is proved for $\epsilon\mathrm{Le} < 1$, stress-free boundary conditions on velocity, and Dirichlet conditions on temperature and solute.

In the case $\epsilon\mathrm{Le} > 1$ and for some values of the parameters $\mathrm{Da}, \mathrm{R}, \mathrm{C}$, we can prove numerically the appearance of overstability (see Fig. 1). It is hence expected that for $\epsilon\mathrm{Le} > 1$ there is a threshold for C and T above which SPES does not hold. We have not been able to prove analytically also this fact, which remains an open problem.

3.3 Numerical investigation

In view of a comparison with the non-linear results of section 4, we consider the case in which instability arises as stationary convection. Under such hypothesis system (6) becomes

$$\begin{cases} 0 = -Z + \mathrm{Da}\,(D^2 - a^2)\,Z + \mathrm{T}\,DW \\ 0 = -a^2(\,\Theta - \Gamma) - (D^2 - a^2)\,W + \mathrm{Da}\,(D^2 - a^2)^2 W - \mathrm{T}\,DZ \\ 0 = \mathrm{R}^2\,W + (D^2 - a^2)\,\Theta \\ 0 = \mathrm{LeC}^2\,W + (D^2 - a^2)\,\Gamma. \end{cases} \tag{10}$$

A few considerations on (10) can be made immediately: the system does not depend on ϵ and depends on Le and C^2 only through LeC^2. It follows that its solutions depend only on three independent physical quantities, i.e. T, Da, and LeC^2 (and on the parameters $\alpha_\pm, \alpha'_\pm$ that appear in the boundary conditions). Furthermore we note that if the boundary conditions on Θ and Γ are the same (i.e. $\alpha_\pm = \alpha'_\pm$) then the system becomes equivalent to a system whose two last equations are replaced with $0 = (\mathrm{R}^2 - \mathrm{LeC}^2)W + (D^2 - a^2)\,(\Theta - \Gamma)$. The resulting system is formally equivalent to the one describing an homogeneous fluid (with the substitutions $\Theta \to \Theta - \Gamma, \mathrm{R}^2 \to \mathrm{R}^2 - \mathrm{LeC}^2$). It follows that the critical Rayleigh number equals that of the homogeneous case translated by LeC^2 and hence the dependence of the solutions on C^2 is trivial. Under the same hypotheses on boundary conditions, from the last two equations we can also derive $(D^2 - a^2)\,(\mathrm{LeC}^2\,\Theta - \mathrm{R}^2\,\Gamma) = 0$, which implies $\mathrm{LeC}^2\,\Theta - \mathrm{R}^2\,\Gamma = 0$ except for a discrete set of values of a.

In all the plots that follow, we keep equal the parameters relative to temperature and solute concentration and those relative to the upper and lower boundaries as well, i.e. $\alpha_+ = \alpha_- = \alpha'_+ = \alpha'_- \equiv \alpha$. The analysis of independent variations of $\alpha_+, \alpha_-, \alpha'_+, \alpha'_-$, in correspondence with fixed values of the other parameters, would require a much longer investigation. We illustrate the effects of the independent variation of $\mathrm{Da}, \mathrm{T}^2$ and of the transition of boundary condition from fixed temperatures and solute concentrations ($\alpha = 0$) to fixed heat flux and solute flux ($\alpha = 1$). Computations are performed for either stress free (F) or rigid (R) conditions on both boundaries.

Fig. 2 illustrates clearly that rotation has stabilizing effects for every value of α, while the most stabilizing conditions are those of fixed temperature and solute concentration. Transition from Dirichlet to Neumann boundary conditions is destabilizing. A partial analysis of the case $\alpha_+ \neq \alpha_-$, which we do not present in this article, shows that R_c^2 and a_c are monotonically increasing with respect to α_+ or α_- independently, while they are monotonically decreasing with respect to $\alpha'_\pm$.

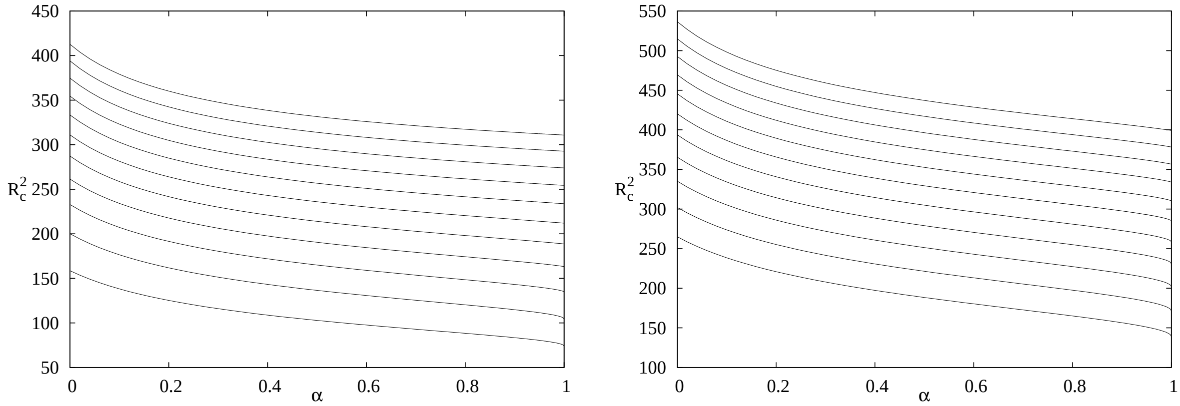

Figure 2: Critical R^2 as function of α when $\mathrm{T}^2 = 0, 5, 10, \ldots, 50$ from bottom to top. Left FF boundaries, right RR boundaries.

In Fig. 3 we illustrate the much more complex stabilizing effect of Darcy's coefficient. Also in this case Neumann boundary conditions on temperature and solute are the most destabilizing. On the other hand the dependence of critical Rayleigh number R_c^2 on Da is not monotonous. This is more evident in the stress-free case. Moreover, the sensitivity of R_c^2 on the boundary conditions (α) becomes more pronounced as Da increases, i.e. when the effective viscosity μ_2 increases.

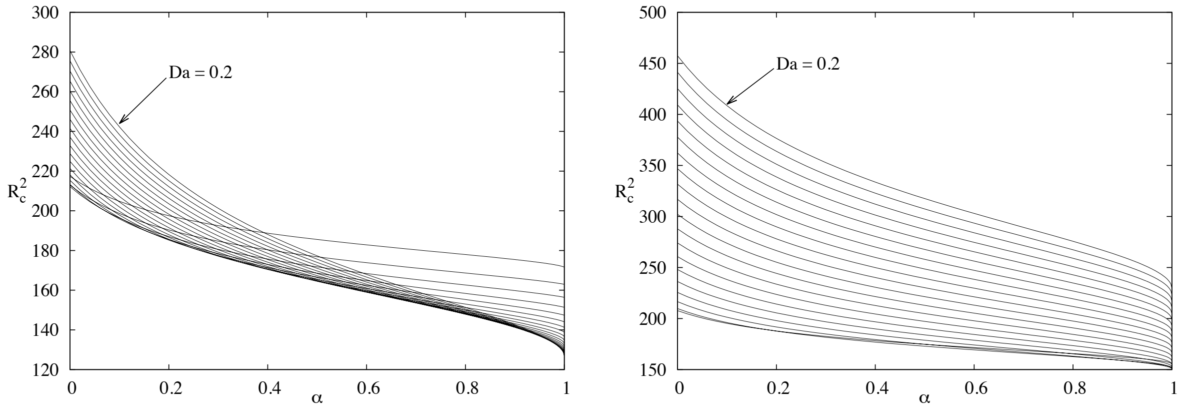

Figure 3: Critical R^2 as function of α when $\mathrm{Da} = 0.01, 0.02, \ldots, 0.2$ from bottom to top. Left FF boundaries, right RR boundaries.

The transition from fixed temperature and solute to fixed heat and solute fluxes has a much stronger effect on the wave number of the critical perturbation. Indeed, as was noted both in fluid dynamics and in flows in porous media (see e.g. [8, 9]), when the boundary conditions approach fixed heat flux, the critical wave number can go to zero (i.e. the wavelength tends to infinity). In Fig. 4 (left panel) we show how, for T small the critical wave number goes to zero when approaching fixed flux boundary conditions while for T big enough the critical wave number tends to a non-zero value when α tends to 1. The critical threshold T_a such that if $\mathrm{T} < \mathrm{T}_a$ then $a_c = 0$ for a given Da number is shown in Fig. 4 (right panel) in stress-free and rigid cases. From this last plot it is clear that there exists a Taylor number below which the critical wave number is zero for every value of Da.

In principle, one should expect no differences between rigid and stress-free cases when the Darcy number is zero. In fact in this case the equations become of second order in W, and the conditions on the value of W should suffice to determine the solution. This does not happen in our plots, and will require further investigations.

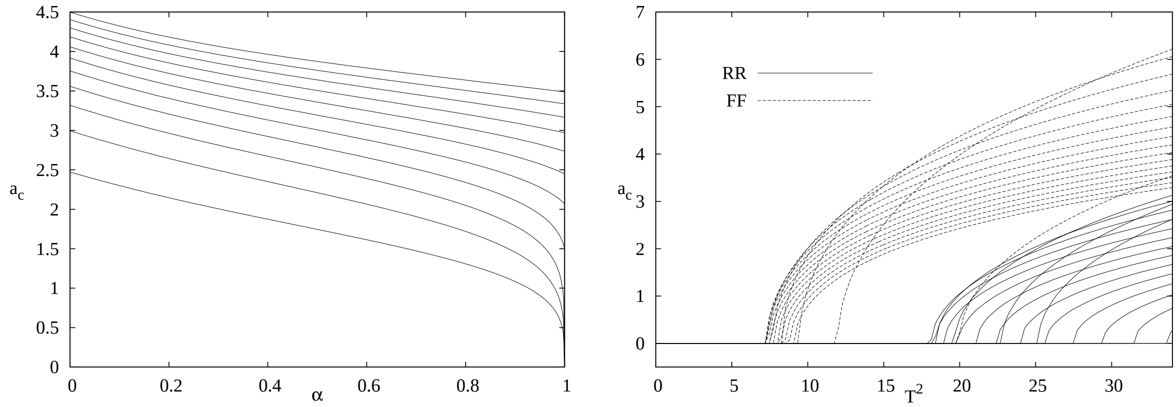

Figure 4: Left: critical wave number a_c as function of α, with $\mathrm{Da} = 0.1$ and $\mathrm{T}^2 = 0, 5, 10, \dots, 50$ from bottom to top. When $\alpha = 1$, a_c is non-zero for T above a threshold. Right: dependence of a_c on T for fixed heat fluxes; $\mathrm{Da} = 0, 0.005, \dots, 0.08$.

The remarkable behavior of the critical wave number is better represented in Fig. 5, where the boundary conditions are of fixed heat and solute flow, and some values of T are chosen big enough to have a positive a_c for some Da numbers. In this case there exist an interval of values of the Darcy number for which the critical wave number is non-zero. Such interval grows monotonically with T. This effect seems to indicate a complex interaction of the two viscosity coefficients and rotation, whose physical interpretation escapes our understanding.

4 Nonlinear stability via energy method

It is a known fact that using the classical energy $(\|\vartheta\|^2 + \epsilon \|\gamma\|^2)/2$ as Lyapunov function, the stabilizing gyroscopic effects, due to solute gradient and rotation, are lost. To build an optimal Lyapunov function, we need to introduce a different "generalized" energy. In this article we obtain two Lyapunov functions using two different field transformations. In section 4.1 we use the classical canonical reduction method (see [28], [31], [19]), which relies on the eigenvalues of the linear problem. In section 4.2 we use the energy based on the field transformation proposed by [29] for fluids.

4.1 First Lyapunov function

The transformation proposed in [31] (in the absence of rotation) is a transformation that typically depends on the wave number of the linear solution which, in the stress free case can be found analytically. In other cases the method requires the use of numerical solutions of the linear system.

Nevertheless, when $\epsilon\mathrm{Le} = 1$ the transformation depends only on the parameters $\mathrm{Le}, \mathrm{C}, \mathrm{R}$, and knowledge of the linear solutions is not required. Under this hypothesis, and applying the substitutions $\theta \to \mathrm{R}\,\theta$

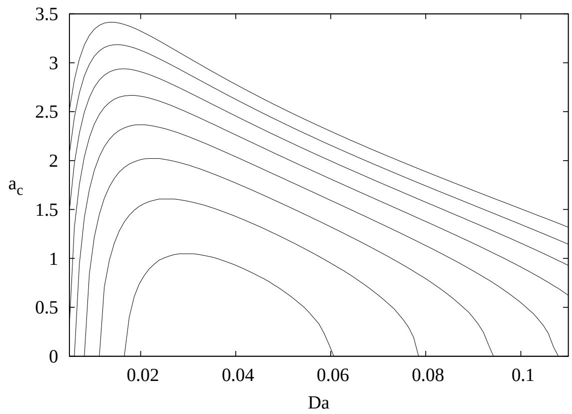

Figure 5: Critical wave number a_c as function of Da for fixed heat flux and $\mathrm{T}^2 = 8, 9, \ldots, 15$ from bottom to top.

and $\gamma \to \mathrm{C}\,\gamma$, system (4) becomes

$$\begin{cases} -\zeta + \mathrm{Da}\,\Delta\zeta + \mathrm{T}\,w_z = 0 \\ \Delta^*(\mathrm{R}^2\,\theta - \mathrm{LeC}^2\,\gamma) - \Delta w + \mathrm{Da}\,\Delta\Delta w - \mathrm{T}\,\zeta_z = 0 \\ \theta_t + \mathbf{u}\cdot\nabla\theta = w + \Delta\theta \\ \gamma_t + \mathrm{Le}\,\mathbf{u}\cdot\nabla\gamma = w + \Delta\gamma. \end{cases}$$

The field transformation and its inverse are

$$\begin{cases} \phi = \frac{1}{\mathrm{R}^2-\mathrm{LeC}^2}(\mathrm{R}^2\theta - \mathrm{LeC}^2\gamma) \\ \psi = \frac{\mathrm{R}^2}{\mathrm{R}^2-\mathrm{LeC}^2}(\theta-\gamma), \end{cases} \qquad \begin{cases} \theta = \phi + \frac{\mathrm{LeC}^2}{\mathrm{R}^2}\psi \\ \gamma = \phi + \psi, \end{cases} \tag{11}$$

the corresponding equations in the new fields become

$$\begin{cases} 0 = -\zeta + \mathrm{Da}\,\Delta\zeta + \mathrm{T}\,w_z \\ 0 = -\Delta w + (\mathrm{R}^2 - \mathrm{LeC}^2)\,\Delta^*\phi + \mathrm{Da}\,\Delta\Delta w - \mathrm{T}\,\zeta_z \\ \psi_t = \frac{\mathrm{R}^2(1-\mathrm{Le})}{\mathrm{R}^2-\mathrm{LeC}^2}\mathbf{u}\cdot\nabla\phi - \frac{\mathrm{Le}(\mathrm{R}^2-\mathrm{C}^2)}{\mathrm{R}^2-\mathrm{LeC}^2}\mathbf{u}\cdot\nabla\psi + \Delta\psi \\ \phi_t = -\frac{\mathrm{R}^2-\mathrm{Le}^2\,\mathrm{C}^2}{\mathrm{R}^2-\mathrm{LeC}^2}\mathbf{u}\cdot\nabla\phi - \frac{\mathrm{LeC}^2(1-\mathrm{Le})}{\mathrm{R}^2-\mathrm{LeC}^2}\mathbf{u}\cdot\nabla\psi + \Delta\phi + w. \end{cases} \tag{12}$$

The boundary conditions for the new fields ϕ, ψ are not decoupled unless θ and γ are subject to the same conditions, i.e. $\alpha_\pm = \alpha'_\pm$, in which case they are

$$\begin{array}{ll} \alpha_\pm\phi_z \pm (1-\alpha_\pm)\phi = 0 & \text{on } z = \pm 1/2 \\ \alpha_\pm\psi_z \pm (1-\alpha_\pm)\psi = 0 & \text{on } z = \pm 1/2. \end{array} \tag{13}$$

We introduce the following generalized energy

$$E_0 = \frac{1}{2}\big(\|\psi\|^2 + \|\phi\|^2\big). \tag{14}$$

Despite its independence on ζ, w, this function can be proven to be zero, in the manifold given by the first two equations, only in the origin. The time derivative of E, according to equations (12), is $\dot{E}_0 = F_2 + F_3$ where

$$\begin{aligned} F_2 &= (\phi, w) + (\phi, \Delta\phi) + (\psi, \Delta\psi), \\ F_3 &= \tfrac{\mathrm{R}^2+\mathrm{LeC}^2}{\mathrm{R}^2-\mathrm{LeC}^2}(1-\mathrm{Le})(\mathbf{u}\cdot\nabla\phi, \psi). \end{aligned} \tag{15}$$

When the boundary conditions for solute concentration and temperature are the same, then F_2 can be rewritten as

$$F_2 = (\phi, w) - \|\nabla\phi\|^2 - S(\phi, \alpha_+, \alpha_-) - \|\nabla\psi\|^2 - S(\psi, \alpha_+, \alpha_-).$$

When $\mathrm{Le} = 1$ the nonlinear term disappears, and the stability condition is *global*. When $\mathrm{Le} \neq 1$ we must take into consideration also the non linear contribution F_3, as described in [28].

To take into account the functional constraints provided by the first two equations, we multiply them respectively by ζ and w and integrate over Ω, obtaining

$$\begin{aligned} V_1 &= -\|\zeta\|^2 - \mathrm{Da}\,\|\nabla\zeta\|^2 + \mathrm{T}\,(w_z, \zeta) = 0 \\ V_2 &= (\mathrm{R}^2 - \mathrm{LeC}^2)(w, \Delta^*\phi) + \|\nabla w\|^2 + \mathrm{Da}\,\|\Delta w\|^2 + \mathrm{T}\,(w_z, \zeta) = 0. \end{aligned} \tag{16}$$

Note that equation $(16)_1$ implies $(w_z, \zeta) \geq 0$, then from $(16)_2$ we get also $(\mathrm{R}^2 - \mathrm{LeC}^2)(w, \Delta^*\phi) < 0$, or $(\mathrm{R}^2 - \mathrm{LeC}^2)(w, \phi) > 0$. Then in the region $\mathrm{R}^2 < \mathrm{LeC}^2$ of parameter space, it is necessarily $(w, \phi) < 0$, and hence $F_2 < 0$, so that the Lyapunov function (14) proves at least linear stability of the equilibrium.

We are left to study the stability in the case $\mathrm{R}^2 > \mathrm{LeC}^2$, i.e. show that $\dot{E}_0$ is negative in a punched neighborhood of the origin. To do so, we need to begin by investigating the signature of the quadratic function F_2.

The function F_2, subject to the constraints V_1, V_2, can be expressed introducing Lagrange multipliers λ_1, λ_2. The resulting function can be decomposed as $F_2 = I - D$, where

$$\begin{aligned} I &= (w, \phi) + (\lambda_1 + \lambda_2)\mathrm{T}(w_z, \zeta) + \lambda_2(\mathrm{R}^2 - \mathrm{LeC}^2)(w, \Delta^*\phi) + \\ &\quad -\lambda_1(\|\zeta\|^2 + \mathrm{Da}\,\|\nabla\zeta\|^2) \\ D &= -(\phi, \Delta\phi) - (\psi, \Delta\psi) - \lambda_2(\|\nabla w\|^2 + \mathrm{Da}\,\|\Delta w\|^2). \end{aligned} \tag{17}$$

Observe that $D > 0$ whenever $\lambda_2 < 0$. From $V_1 = 0$ it follows

$$\mathrm{Da}\,\|\nabla\zeta\|^2 = -\|\zeta\|^2 + \mathrm{T}(w_z, \zeta) \leq \mathrm{T}(w_z, \zeta) \leq T\,\|w\|\,\|\zeta_z\| \leq T\,\|w\|\,\|\nabla\zeta\|,$$

i.e. $\|\nabla\zeta\| \leq T\,\|w\|/\mathrm{Da}$. Using this inequality, one can show that the ratio I/D is bounded from above. We can define then $m = \max(I/D)$, where the maximum is taken over the space $\mathcal{H}$ of functions satisfying boundary conditions and constraints $(12)_{1,2}$. F_2 is then negative definite in the region of parameter space in which $m < 1$, where in fact

$$F_2 = \left(\frac{I}{D} - 1\right) D \leq \left(\max_{\mathcal{H}}\left[\frac{I}{D}\right] - 1\right) D = (m-1)D \tag{18}$$

holds. We hence need to determine the critical points $\bar{\sigma} = (\bar{\zeta}, \bar{w}, \bar{\phi}, \bar{\psi})$ which are critical for the function I/D and such that $I(\bar{\sigma})/D(\bar{\sigma}) = 1$. This condition corresponds to the equations

$$\begin{cases} (\lambda_2 - \lambda_1)\,\mathrm{T}\,\bar{w}_z = 0 \\ \bar{\phi} - \lambda_1\,\mathrm{T}\,\bar{\zeta}_z + \lambda_2\,(\mathrm{Da}\,\Delta^2\bar{w} - \Delta\bar{w}) = 0 \\ \bar{w} + 2\,\Delta\bar{\phi} + \lambda_2((\mathrm{R}^2 - \mathrm{LeC}^2)\Delta^*\bar{w}) = 0 \\ \Delta\bar{\psi} = 0, \end{cases} \tag{19}$$

where the first two equations have been simplified using $(12)_1$.

From the first equation one obtains that $\lambda_1 = \lambda_2 = \lambda$ then, from the second equation one obtains that $\bar{\phi} + \lambda(\mathrm{R}^2 - \mathrm{LeC}^2)\Delta^*\bar{\phi} = 0$, from which it follows that $\lambda^{-1} = -a^2(\mathrm{R}^2 - \mathrm{LeC}^2)$. The last two equations become $0 = \bar{w} + \Delta\bar{\phi}$, and $0 = \Delta\bar{\psi}$, which are precisely the equations satisfied by a marginal convective solution of system (12). This indicates the coincidence of the linear instability and energy stability thresholds.

To estimate the non-quadratic part of $\dot{E}$, we recall equation $(3)_1$

$$\nabla p = (\mathrm{R}\,\theta - \mathrm{C}\,\gamma)\mathbf{k} - \mathbf{u} + \mathrm{Da}\,\Delta\mathbf{u}.$$

Rewriting it according to the substitutions employed in deriving system (12), i.e. $\theta \to \mathrm{R}\,\theta$, $\gamma \to \mathrm{LeC}\,\gamma$, and (11), we obtain

$$\nabla p = (\mathrm{R}^2 - \mathrm{LeC}^2)\phi\,\mathbf{k} - \mathbf{u} + \mathrm{Da}\,\Delta\mathbf{u}.$$

Multiplying by $\mathbf{u}$ and integrating over the cell Ω, we obtain

$$\|\mathbf{u}\|_2^2 + \mathrm{Da}\,\|\nabla\mathbf{u}\|_2^2 = (\mathrm{R}^2 - \mathrm{LeC}^2)(w,\phi).$$

Using Cauchy-Schwarz and Poincaré inequalities for w, we obtain that

$$\begin{aligned}\|\nabla\mathbf{u}\|_2^2 &\le \tfrac{|\mathrm{R}^2-\mathrm{LeC}^2|}{\mathrm{Da}}|(w,\phi)| \le \tfrac{|\mathrm{R}^2-\mathrm{LeC}^2|}{\mathrm{Da}}\,\|w\|_2\,\|\phi\|_2 \le \\ &\le \tfrac{|\mathrm{R}^2-\mathrm{LeC}^2|}{\pi\mathrm{Da}}\,\|\nabla w\|_2\,\|\phi\|_2 \le \tfrac{|\mathrm{R}^2-\mathrm{LeC}^2|}{\pi\mathrm{Da}}\,\|\nabla\mathbf{u}\|_2\,\|\phi\|_2\,,\end{aligned}$$

which can be summarized in the inequality

$$\|\nabla\mathbf{u}\|_2 \le \frac{|\mathrm{R}^2 - \mathrm{LeC}^2|}{\pi\,\mathrm{Da}}\,\|\phi\|_2\,.$$

From which also follows

$$\|\mathbf{u}\|_2 \le \frac{|\mathrm{R}^2 - \mathrm{LeC}^2|}{\pi^2\,\mathrm{Da}}\,\|\phi\|_2\,.$$

To estimate the nonlinear term F_3, one can use Hölder inequality to obtain that $|(\mathbf{u}\cdot\nabla\phi,\psi)| \le \|\mathbf{u}\|_4\,\|\psi\|_4\,\|\nabla\phi\|$. By Sobolev embedding theorems (see [39] page 389), one has that

$$\begin{aligned}\|\mathbf{u}\|_4\,\|\psi\|_4\,\|\nabla\phi\|_2 &\le 32a^{\frac{1}{3}}\sqrt{\|\mathbf{u}\|_2^2 + \|\nabla\mathbf{u}\|_2^2}\sqrt{\|\psi\|_2^2 + \|\nabla\psi\|_2^2}\,\|\nabla\phi\|_2 \le \\ &\le C_1\,\|\phi\|_2\,C_2\sqrt{(\psi,\Delta\psi)(\phi,\Delta\phi)}\end{aligned} \tag{20}$$

where $C_1 = 32a^{\frac{1}{3}}|\mathrm{R}^2 - \mathrm{LeC}^2|\sqrt{1+\pi^2}\,\pi^{-2}\mathrm{Da}^{-1}$ and $C_2 = \sqrt{1+\xi^2}$. ξ is a constant coming from the generalized Poincaré inequality

$$\|\psi\|_2 \le \xi\sqrt{-(\psi,\Delta\psi)}$$

for functions satisfying Robin boundary conditions (13), whose dependency on the parameters α_+,α_- is shown in Fig. 6 in the particular case $\alpha_+ = \alpha_- = \alpha$.

Using the fact that $\|\phi\| \le \sqrt{2E}$ and the definition of D, we then obtain that

$$\begin{aligned}C_1C_2\,\|\phi\|_2\,\sqrt{(\psi,\Delta\psi)(\phi,\Delta\psi)} &\le C_1C_2\sqrt{2}E^{1/2}\tfrac{1}{2}(-(\phi,\Delta\phi) - (\psi,\Delta\psi)) \\ &\le \tfrac{\sqrt{2}}{2}C_1C_2E^{1/2}D.\end{aligned} \tag{21}$$

Summarizing, we have shown that

$$|F_3| \le \frac{\sqrt{2}}{2}C_1C_2\left|\frac{(\mathrm{R}^2+\mathrm{LeC}^2)(1-\mathrm{Le})}{\mathrm{R}^2-\mathrm{LeC}^2}\right|E^{1/2}D. \tag{22}$$

It hence follows from (18) and (22) that $\dot{E} \le \chi D$, where

$$\chi = m - 1 + \frac{\sqrt{2}}{2}C_1C_2\left|\frac{(\mathrm{R}^2+\mathrm{LeC}^2)(1-\mathrm{Le})}{\mathrm{R}^2-\mathrm{LeC}^2}\right|E^{1/2}. \tag{23}$$

Whenever the physical parameters are below threshold, which is equivalent to the fact that $m-1<0$, all initial data such that

$$E \le 2(m-1)^2\frac{(\mathrm{R}^2-\mathrm{LeC}^2)^2}{(\mathrm{R}^2+\mathrm{LeC}^2)^2(1-\mathrm{Le})^2\,C_1^2\,C_2^2} \tag{24}$$

yield a negative χ. The generalized Poincaré inequality recalled above gives $\dot{E} \le \chi\,D \le 2\,\chi\,\xi^{-2}\,E$. It hence follows that

Theorem 1 *Let $\sigma(t,x,y,z) = (\zeta,w,\phi,\psi)(t,x,y,z)$ be any solution of equation (12) with physical parameters such that $m<1$, and denote by $E(t) = E(\sigma(t,x,y,z))$. If σ is such that $E(0)$ satisfies equation (24), then $E(t) \le e^{2\chi\xi^{-2}t}E(0)$ where χ is the negative constant defined in (23). This implies that the basic motion is conditionally asymptotically exponentially Lyapunov stable.*

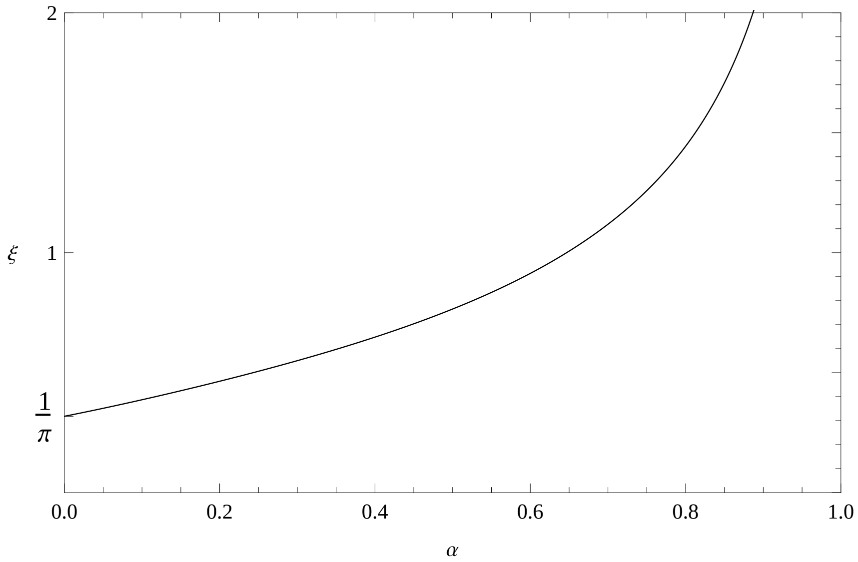

Figure 6: Poincaré constant for functions with Robin boundary conditions

4.2 Second Lyapunov function

For $\epsilon\mathrm{Le} \neq 1$ we use the transformation introduced for fluid dynamics in [29] and used also in [18]. We consider again system (4) and introduce the auxiliary fields

$$\begin{cases} \phi = \mathrm{R}\,\theta - \mathrm{LeC}\,\gamma \\ \psi = \mathrm{R}\,\theta - \delta\,p\,\mathrm{LeC}\,\gamma \end{cases} \quad \leftrightarrow \quad \begin{cases} \theta = \frac{1}{\mathrm{R}(\delta\,p-1)}(\delta\,p\,\phi - \psi) \\ \gamma = \frac{1}{\mathrm{LeC}(\delta\,p-1)}(\phi - \psi) \end{cases}$$

where p denotes $\epsilon\mathrm{Le}$, and we assume $\delta p \neq 1$. Making the above substitution and linearizing, we obtain

$$\begin{cases} -\zeta + \mathrm{Da}\,\Delta\zeta + \mathrm{T}\,w_z = 0 \\ \Delta^*\phi - \Delta w + \mathrm{Da}\,\Delta\Delta w - \mathrm{T}\,\zeta_z = 0 \\ \psi_t = (\mathrm{R}^2 - \delta\,\mathrm{LeC}^2)w + \frac{1}{\delta\,p-1}\big[\delta\,(p-1)\Delta\phi + (\delta-1)\Delta\psi\big] \\ p\phi_t = (p\,\mathrm{R}^2 - \mathrm{LeC}^2)w + \frac{1}{\delta\,p-1}\big[(\delta\,p^2-1)\Delta\phi + (1-p)\Delta\psi\big]. \end{cases}$$

In this case, when $p = 1$ we obtain global stability. We introduce then the Lyapunov function (generalized energy) $E = \frac{1}{2}\big(\mu\,\|\psi\|^2 + p\,\|\phi\|^2\big)$, with μ a (positive) parameter. By multiplying the last two equations just found by ψ and ϕ respectively, and evaluating the time derivative $\dot{E}$ of the energy, we obtain that

$$\dot{E} = \mu(\mathrm{R}^2 - \delta\,\mathrm{LeC}^2)(w,\psi) + (p\,\mathrm{R}^2 - \mathrm{LeC}^2)(w,\phi) - C(\phi,\psi),$$

where $C(\phi,\psi)$ in the function

$$\begin{aligned} C(\phi,\psi) &= -c_1(\psi,\Delta\psi) - c_2(\phi,\Delta\psi) - c_3(\phi,\Delta\phi) = \\ &c_1\,\|\nabla\psi\|^2 + c_2(\nabla\phi,\nabla\psi) + c_3\,\|\nabla\phi\|^2 + \\ &\quad + \tfrac{1-\alpha_+}{\alpha_+}\left(c_1\psi_+^2 + c_2\phi_+\psi_+ + c_3\phi_+^2\right) + \tfrac{1-\alpha_-}{\alpha_-}\left(c_1\psi_-^2 + c_2\phi_-\psi_- + c_3\phi_-^2\right), \end{aligned}$$

where

$$c_1 = \mu\frac{\delta-1}{\delta\,p-1},\ c_2 = \frac{(\mu\,\delta-1)(p-1)}{\delta\,p-1},\ c_3 = \frac{\delta\,p^2-1}{\delta\,p-1}.$$

To take into account the first two equations, we multiply them respectively by ζ and w, integrate over Ω

$$\begin{cases} -\|\zeta\|^2 - \mathrm{Da}\,\|\nabla\zeta\|^2 + \mathrm{T}\,(w_z,\zeta) = 0 \\ (w,\Delta^*\phi) + \|\nabla w\|^2 + \mathrm{Da}\,\|\Delta w\|^2 + \mathrm{T}\,(w_z,\zeta) = 0. \end{cases}$$

We multiply the above equations by the new parameters λ_1 and λ_2 and add them to the expression previously obtained for $\dot{E}$.

For $C(\phi,\psi)$ to be positive definite, we need to impose the two conditions $c_1 > 0, c_2^2 - 4c_1c_3 < 0$, which can be viewed as conditions on δ and μ. The time derivative of E can be then expressed as $\dot{E}$ as $\dot{E} = I - D$ where

$$\begin{aligned} I = \quad & \mu(\mathrm{R}^2 - \delta\,\mathrm{LeC}^2)(w,\psi) + (p\,\mathrm{R}^2 - \mathrm{LeC}^2)(w,\phi) + \lambda_2(w,\Delta^*\phi) \\ & +(\lambda_1+\lambda_2)\mathrm{T}\,(w_z,\zeta) - \lambda_1\left(\|\zeta\|^2 + \mathrm{Da}\,\|\nabla\zeta\|^2\right) \\ D = \quad & \lambda_2\left(\|\nabla w\|^2 + \mathrm{Da}\,\|\Delta w\|^2\right) + C(\phi,\psi). \end{aligned} \tag{25}$$

We have then

$$\dot{E} \le \left(\max_{\mathcal{H}}\left[\frac{I}{D}\right] - 1\right) D.$$

We derive then the Euler-Lagrange equations associated to the above maximum problem. The equation for the variation of ζ is

$$(\lambda_1+\lambda_2)\,T\,w_z - 2\lambda_1\,\zeta + 2\lambda_1\,\mathrm{Da}\,\Delta\zeta = 0,$$

from which, using the first equation of motion, it follows that $\lambda_1 = \lambda_2 = \lambda$. The other Euler-Lagrange equations are

$$\begin{cases} \mu(\mathrm{R}^2 - \delta\,\mathrm{LeC}^2)\psi + (p\,\mathrm{R}^2 - \mathrm{LeC}^2 + \lambda\,a^2)\phi = 0 \\ (p\,\mathrm{R}^2 - \mathrm{LeC}^2 - \lambda\,a^2)w + 2\,c_3\,\Delta\phi + c_2\,\Delta\psi = 0 \\ \mu(\mathrm{R}^2 - \delta\,\mathrm{LeC}^2)w + c_2\,\Delta\phi + 2\,c_1\,\Delta\psi = 0 \end{cases} \tag{26}$$

with c_1, c_2, c_3 defined previously and where the second equation was simplified using $(12)_1$. The parameters λ, μ, δ, that we call Lyapunov parameters, define a family of candidate Lyapunov functions. Fixing physical parameters which give linear stability there are appropriate choices of these parameters which provide a Lyapunov function. Indeed, our numerical investigation shows that there is a 1-parametric choice of such parameters. This choice of Lyapunov functions proves coincidence of the threshold of linear instability with that of energy stability.

The numerical procedure is the following. After fixing the physical parameters $\mathrm{Da}, \mathrm{T}, \mathrm{C}, \epsilon, \mathrm{Le}$ and the boundary parameters $\alpha_\pm, \alpha'_\pm$, equations (26) become a spectral eigenvalue problem in R depending on a and the Lyapunov parameters λ, μ, δ. Minimizing with respect to a, we obtain the critical Rayleigh number associated to such Lyapunov parameters. Maximizing then with respect to the parameters we obtain the best choice of Lyapunov function. Comparing the threshold associated to such Lyapunov function with the linear threshold, we see their coincidence.

5 Conclusions

We have analyzed the thermal stability of a binary fluid in a rotating porous layer. The fluid motion is modeled by the Darcy-Brinkman equation and the fluid is subject to several combinations of boundary conditions. We considered in particular the effect of Robin boundary condition on temperature and solute, adapting to this particular setup the two Lyapunov functions introduced in [18, 31]. We obtained coincidence of linear and non linear critical values, and we have shown, in some cases, the global stability of the solutions.

We have shown that the most stabilizing conditions are those with prescribed values of the fields (Dirichlet boundary conditions). The same effect is also described in [8, 9] both for fluids and porous media. In the limit case of fixed heat fluxes, the critical wave number strongly depends on Da and T numbers, and vanishes for any Darcy number for low rotation, and conditionally on Darcy number for high rotation.

This article extends in a non-trivial way the results of [18] who obtain results only under ideal stress-free boundary conditions and fixed temperature and solute concentration at the boundaries.

The numerical investigation confirms the physically relevant appearance of zero wave numbers in some regions of the parameter space of the problem when the boundary conditions on temperature are of fixed flux.

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