Marginal regions for the solute Bénard problem with many types of boundary conditions

P. Falsaperla, Andrea Giacobbe


Abstract

A large number of variants of the Bénard problem (with a solute, rotating, subject to magnetic field, etc) have been extensively studied. Despite this, new interesting results can be obtained imposing very general yet physically relevant boundary conditions. In this framework, we develop a technique to analytically compute the marginal region in parameter space.

We investigate the thermal stability of a fluid layer salted from below, subject to finite slip on velocity and Robin conditions on temperature and solute concentration. We write analytical conditions for the onset of stationary convection, obtain simplified formulas for particularly symmetric cases, and draw the associated (convective) marginal regions in some significant cases. Moreover, we describe the analytical conditions for the onset of overstability, and use such equations to numerically draw the associated (overstable) marginal region. We finally perform an asymptotic analysis for small wave numbers.

Keywords: Bénard problem, binary fluids, stability, finite-slip, Newton-Robin.

M.S.C.: 76E15, 76E06, 34B08, 34B09.

1 Introduction

A central topic in fluid-dynamics is the investigation of stability of fluid flows [1]. In this work we deal with an infinite layer of fluid enclosed between two parallel horizontal boundaries, heated and salted from below. This problem, known as solute Bénard problem, has been studied for example in [2, 3], and is relevant in many geophysical applications (see [2] and references therein).

The solute Bénard problem depends on many physical parameters, which are related to properties of the fluid, and to characteristics of the boundaries. In this article, in particular, we assume very general boundary conditions: finite-slip conditions on the velocity, Newton-Robin conditions on the temperature and Robin conditions on the solute concentration. Boundary-slip conditions were introduced since the very beginning in fluid dynamics [4, 5], and they still are part of many modern treatments in the subject [2, 6, 7, 8]. Several experiments and simulations indicate the existence of slip at the wall for a liquid on a weakly bonding surface. Robin conditions were also introduced in the late 19th century, and are the simpler reasonable extension to the ideal hypothesis of fixed value or fixed gradient of the fields, which are difficult to achieve experimentally. Such conditions are also widely employed and are part of the basic literature in the field [2, 9, 10, 11, 12, 13].

The solute Bénard problem admits a motionless solution in which temperature and concentration have constant vertical gradient. A typical method to discuss the stability of a solution, is to investigate its linear instability by computing the equations of its perturbations. Such equations reduce, after some manipulations and under the standard assumption of periodicity in the $x,y$ directions, to a linear system.
depending on three essential non-dimensional parameters: the wave number \( a \), the Rayleigh number \( R \) and the solute Rayleigh number \( C \). The problem also depends on additional parameters appearing in the boundary conditions and on the Prandtl and Schmidt numbers \( p_\vartheta, p_\gamma \).

Our goal is to write analytic equations to describe the region in parameter space where a change of stability can possibly happen; such region is called marginal region. In this work, we show that the marginal region is the zero level-set of analytic functions, that we call marginal functions. Marginal functions associated to the phenomenon of stationary convection and those associated to overstability are derived with slightly different procedures. Moreover the overstable case produces equations and inequalities (the associated marginal regions are what is called semianalytic manifolds). Marginal functions associated to stationary convection, depend on the parameters \( a, R, C \) via algebraic functions (polynomial and radicals) composed with circular and hyperbolic functions, and depend on boundary parameters linearly. Marginal functions associated to overstability, depend also on \( p_\vartheta, p_\gamma \).

The marginal functions turn out to be the determinant of a matrix computed from the linearized system and the boundary conditions. Considering the boundary parameters as accessory, we investigate marginal functions for particular choices which correspond to extremal cases: rigid, stress-free, fixed temperature, fixed heat flux, fixed solute concentration, and fixed flux of solute. We also deduce the asymptotic behavior of the marginal region for vanishing wave number; in particular, we show that the marginal region approaches the axes \( a = 0 \) only for fixed temperature gradient at both boundaries.

The plan of the paper is as follows: in Section 2 we start from the original system of Navier-Stokes equations in the Oberbeck-Boussinesq approximation, and we relate the linear instability of the motionless solution to a system of linear differential equations with boundary conditions. This system is what we investigate in the work. Section 3 is devoted to the theoretical approach that allows to obtain functions whose zero level-set is the marginal region. In Section 4 we apply the technique to our particular system under the hypothesis that the principle of exchange of stabilities holds, we obtain the marginal functions, and we use them to deduce properties and draw pictures of the associated marginal region. In Section 5 we apply the technique to the overstable case, we describe the structure of the marginal functions, and use them to numerically plot the associated marginal region. In Section 6, we finally describe the asymptotic of the marginal region as \( a \) tends to zero.

## 2 Equations and boundary conditions

In a cartesian frame of reference \( Oxyz \) with unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), consider a layer \( \Omega_d = \mathbb{R}^2_{x,y} \times (-d/2, d/2)_z \) of thickness \( d > 0 \), filled with a newtonian fluid, and subject to the gravity field \( \mathbf{g} = -g \mathbf{k} \). In the Oberbeck-Boussinesq approximation, the equations governing the motion of the fluid are given by (see Ref. [2])

\[
\begin{cases}
(u, v, w)_t + (u, v, w) \cdot \nabla (u, v, w) + \frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0} \mathbf{g} = \nu \Delta (u, v, w) \\
\nabla \cdot (u, v, w) = 0 \\
\vartheta_t + (u, v, w) \cdot \nabla \vartheta = \kappa_\vartheta \Delta \vartheta \\
\gamma_t + (u, v, w) \cdot \nabla \gamma = \kappa_\gamma \Delta \gamma
\end{cases}
\]

where \((u, v, w)\) is the velocity, \(\vartheta\) is the temperature, \(\gamma\) is the solute concentration, and \(p\) is the pressure, while \(\nu, \kappa_\vartheta,\) and \(\kappa_\gamma\) are positive constants which represent kinematic viscosity, and thermal and solute diffusivity. The equations are written making the assumption that the fluid density \(\rho\) depends on temperature and solute concentration according to the linear law

\[
\rho(\vartheta, \gamma) = \rho_0 [1 - c_\vartheta (\vartheta - \vartheta_0) + c_\gamma (\gamma - \gamma_0)],
\]

\[\text{where} \quad c_\vartheta, c_\gamma > 0.\]
obtained by a Taylor expansion of the density function about a reference temperature \( \bar{\vartheta} \) and concentration \( \gamma_0 \). Here \( \rho_0 \) is the reference density, and \( c_\vartheta, c_\gamma \) are positive density variation coefficients for temperature and concentration respectively. Gradient, Laplacian and derivative with respect to time are represented respectively by the symbols \( \nabla \), \( \Delta \), and the subscript “\( \tau \)”. Other than the containment condition \( w = 0 \) on both boundaries, a reasonably simple and yet very general set of boundary conditions for the velocity field are the so called finite-slip condition [6, 7], which are

\[
\begin{align*}
\begin{cases}
\alpha_- (\vartheta_z + G_\vartheta) - (\vartheta - \vartheta_{\pm}) = 0, & \text{on } z = -d/2, \\
\alpha_+ (\vartheta_z + G_\vartheta) + (\vartheta - \vartheta_{\pm}) = 0, & \text{on } z = d/2.
\end{cases}
\end{align*}
\]

Here, \( \alpha_{\pm} \) and \( G_\vartheta \) are non-negative real numbers, \( \vartheta_- = \vartheta_0 + G_\vartheta d/2 \), \( \vartheta_+ = \vartheta_0 - G_\vartheta d/2 \) are respectively the temperatures at the lower (\( \vartheta_- \)) and higher (\( \vartheta_+ \)) boundaries and \( \vartheta_0 \) is the reference temperature (observe that the boundary conditions depend on \( \alpha_{\pm} \) and on two other parameters not on three, as it may seem). The limit \( \alpha_{\pm} = 0 \) is the infinite conductivity boundary condition, in which is fixed the value of the temperature at the corresponding boundary, while the limit \( \alpha_{\pm} = \infty \) is the insulating boundary condition [11, 12, 13], with a fixed heat flux \( q \) of norm \( q = G_\vartheta \kappa_\vartheta \) and directed along the \( z \)-axis at the corresponding boundary. The cases in which \( \alpha_{\pm} \) is finite and positive are cases of finite conductivity at the corresponding boundary.

For the solute field, by similar considerations [2], we use boundary conditions depending on both concentration of solute and its normal derivative at the boundary surfaces. Again, to obtain a range of boundary conditions (from fixed concentrations to fixed fluxes of solute) depending on a single parameter \( \alpha_{\pm} \), the state \((u, v, w, \vartheta, \gamma, p)\) with

\[
\bar{u} = \bar{v} = \bar{w} = 0, \quad \bar{v} = \vartheta_0 - G_\vartheta z, \quad \bar{\gamma} = \gamma_0 - G_\gamma z, \quad \bar{p} = p_0 - \rho_0 g z - \frac{\mu g}{\rho_0} (c_\vartheta G_\vartheta - c_\gamma G_\gamma) z^2.
\]

Note that in (1) \( G_\vartheta, G_\gamma \) are the temperature and concentration gradients. The equations which govern the evolution of a disturbance to (1) are

\[
\begin{align*}
\begin{cases}
(U, V, W)_{t} + (U, V, W) \cdot \nabla (U, V, W) + \frac{\nabla \vartheta}{\rho_0} + (c_\vartheta \Theta - c_\gamma \Gamma) g = \nu \Delta (U, V, W), \\
\nabla \cdot (U, V, W) = 0, \\
\Theta_t + (U, V, W) \cdot \nabla \Theta = G_\vartheta W + \kappa_\Theta \Delta \Theta, \\
\Gamma_t + (U, V, W) \cdot \nabla \Gamma = G_\gamma W + \kappa_\gamma \Delta \Gamma,
\end{cases}
\end{align*}
\]
where \((U, V, W), \Theta, \Gamma, \Pi\) are the perturbations to the velocity, temperature, concentration, and pressure fields, respectively.

Following the standard linear instability analysis of Chandrasekhar [1], applying twice the curl operator to the first equation, and then considering only the linear terms of the resulting systems, one obtains

\[
\begin{align*}
\Delta U_t &= -gc_\Theta \Theta_{xx} + gc_\Gamma \Theta_{zz} + \nu \Delta^2 U, \\
\Delta V_t &= -gc_\Theta \Theta_{yy} + gc_\Gamma \Theta_{yz} + \nu \Delta^2 V, \\
\Delta W_t &= g \Delta^* (c_\Theta \Theta - c_\Gamma \Gamma) + \nu \Delta^2 W, \\
\Theta_t &= G_\Theta W + \kappa_\Theta \Delta \Theta, \\
\Gamma_t &= G_\Gamma W + \kappa_\Gamma \Delta \Gamma,
\end{align*}
\]

(2)

where \(\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2\). The last three equations are autonomous, and the system is solved by determining the solution fields \(W, \Theta, \Gamma\) and then substituting them in the first two equations, which can always be solved [1, II §10]. The boundary condition for the three functions \(W, \Theta, \Gamma\) are

\[
\begin{align*}
W &= 0, \quad \partial_x W = 0, \quad \Theta - \alpha_\Theta \partial_x \Theta = 0, \quad \Gamma - \beta_\Gamma \partial_x \Gamma = 0, \\
W &= 0, \quad \partial_x W = 0, \quad \Theta - \alpha_\Theta \partial_x \Theta = 0, \quad \Gamma + \beta_\Gamma \partial_x \Gamma = 0,
\end{align*}
\]

(3)

respectively on \(z = \pm d/2\). As a last step assume, as usual, that the perturbation fields are sufficiently smooth, and that they are periodic in the \(x\) and \(y\) directions. Substituting the functions \(W, \Theta, \Gamma\) in (2) with functions of the form

\[W(z)e^{i(k_x x + k_y y)}e^{(\sigma + i \tau) t}, \quad \Theta(z)e^{i(k_x x + k_y y)}e^{(\sigma + i \tau) t}, \quad \Gamma(z)e^{i(k_x x + k_y y)}e^{(\sigma + i \tau) t}\]

(with some abuse of notation), and denoting by \(k = (k_x^2 + k_y^2)^{1/2}\) the wave number, one obtains the following system of ODE for the perturbation fields \(W, \Theta, \Gamma\), which are now functions of \(z\) only and where \(D\) is the derivation with respect to \(z\)

\[
\begin{align*}
(\sigma + i \tau)(D^2 - k^2)W &= -gk^2 (c_\Theta \Theta - c_\Gamma \Gamma) + \nu (D^2 - k^2)^2 W, \\
(\sigma + i \tau) \Theta &= G_\Theta W + \kappa_\Theta (D^2 - k^2) \Theta, \\
(\sigma + i \tau) \Gamma &= G_\Gamma W + \kappa_\Gamma (D^2 - k^2) \Gamma.
\end{align*}
\]

(4)

A change in the stability of the purely conducting solution can take place only for a choice of parameters for which there exists a solution to the equations (4), (3) in which the real part \(\sigma\) of the eigenvalue is equal to zero. It is hence possible to explicitly define the marginal region as the subset of parameter space in which there exist non-zero solutions of equations (4), (3) with \(\sigma = 0\).

By posing \(dz_1 = z, \ a = kd, \ W_1 = \nu W/d^2, \ \Theta_1 = gc_\Theta \Theta, \ \Gamma_1 = gc_\Gamma \Gamma, \ \tau_1 = \tau d^2/\nu\), introducing \(R^2 = gc_\Theta G_\Theta d^4/(\nu k_\Theta), \ C^2 = gc_\Gamma G_\Gamma d^4/(\nu k_\Gamma)\) (the Rayleigh number and the solute Rayleigh number respectively), \(p_\Theta = \nu/k_\Theta, \ p_\Gamma = \nu/k_\Gamma\) (the Prandl and Schmidt numbers), and eliminating the subscript from \(z_1, W_1, \Theta_1, \Gamma_1, \tau_1\), equations (4) can be cast in the non-dimensional form

\[
\begin{align*}
i \tau (D^2 - a^2) W &= -a^2 \Theta + a^2 \Gamma + (D^2 - a^2)^2 W, \\
i \tau p_\Theta \Theta &= R^2 W + (D^2 - a^2) \Theta, \\
i \tau p_\Gamma \Gamma &= C^2 W + (D^2 - a^2) \Gamma.
\end{align*}
\]

(5)

The set of points of the parameter space for which there exist non-zero solutions to equations (5) satisfying boundary conditions

\[
W = 0, \quad DW \pm \lambda \pm D^2 W = 0, \quad \Theta \pm \alpha \pm D \Theta = 0, \quad \Gamma \pm \beta \pm D \Gamma = 0
\]

(6)

on \(z = \pm 1/2\) (with \(\lambda, \alpha, \beta\) substituted by their non-dimensional version), is the object of our investigation.

The marginal region consists of two hypersurfaces in parameter space: that associated to the existence of solutions to the above equations with \(\tau = 0\), called convective marginal region, and that associated to
solutions with $\tau \neq 0$, called overstable marginal region. Naturally, the values of parameters that separate linear stability from instability are a subset of the marginal region.

The original contribution of this article consists in: calculating analytic functions whose zero level-set is the convective marginal region (convective marginal functions); using the functions to obtain information on such region; describing the functions associated to overstability (overstable marginal functions); using them to draw the overstable marginal region; describing the asymptotic of the marginal region for the wave number $a$ close to zero.

We postpone computation and investigation of the marginal functions to Section 4–6, while we devote Section 3 to the technique that allows us to compute such functions.

2.1 On marginal regions

The linearized P.D.E. (2) we are investigating, can be thought as a linear operator from an appropriate Hilbert space. The spectral analysis of such operator corresponds to the existence of non-zero solutions to equations (4), (3) and a choice of parameters is called marginal if the spectrum of the operator associated to such parameters contains eigenvalues with zero real part. In Sections 4–5, we prove that the subset of parameter space for which zero is an eigenvalue of the operator — the convective marginal region — is the zero level-set of a function, while the subset of parameter space such that $\pm i\tau$ are eigenvalues of the operator for some $\tau$ positive real — the overstable marginal region — is the union over $\tau$ of the zero level-set of two $\tau$-dependent functions. Both subsets are codimension 1 submanifolds of the parameter space.

Paradigmatic is the simple answer to the following question: given a parameter space $\mathbb{R}^m$, and a family of $\lambda$-dependent $n \times n$ matrices $M_\lambda$, what is the subspace of parameter space for which $M_\lambda$ has eigenvalues with zero real part?

Calling marginal region such subspace, the answer to the question is the following. The marginal region is the union of two subsets: the set of parameters for which $M_\lambda$ has zero as eigenvalue, and the set of parameters for which $M_\lambda$ has the two purely imaginary eigenvalues $\pm i\tau$. To stress the connection with our problem, we call such regions convective marginal region and overstable marginal region respectively.

The convective marginal region is simply the zero level-set of the function $\det(M_\lambda)$. The overstable marginal region is less straightforward to describe. In fact, the remainder of the division of the characteristic polynomial of $M(\lambda)$ by $x^2 + \tau^2$ is a polynomial of the type $p(\lambda; \tau) + q(\lambda; \tau)x$. A point $\lambda$ belongs to the overstable marginal region if and only if $\lambda$ is such that $p(\lambda; \tau) + q(\lambda; \tau) = 0$ for some $\tau$. It follows that for fixed $\tau$ the level-set $\Lambda_\tau = \{\lambda \mid p(\lambda; \tau) + q(\lambda; \tau) = 0\}$ is a codimension 2 submanifold, and the overstable marginal region is the codimension 1 submanifold obtained taking the union of $\Lambda_\tau$ as $\tau$ varies among the positive reals.

Simple yet interesting examples can be found investigating $2 \times 2$ matrices depending on two parameters. The convective marginal region corresponds to the set $\det(M) = 0$, while the overstable marginal region corresponds to the set $\{\tau(M) = 0, \det(M) > 0\}$. These two conditions yield captivating pictures whose main feature is that the convective marginal region is an algebraic curve, while the overstable marginal region is a semi-algebraic curve that stops when hitting the convective marginal region: a feature we will encounter in Section 5. In Figure 1 two plots relative to the two sample matrices (unrelated to physical problems)

$$M_1 = \begin{bmatrix} a - 2b - 2 & a + 2b + 1 \\ b - a & 2b \end{bmatrix}, \hspace{1cm} M_2 = \begin{bmatrix} a^2 - 3ab & 3a^2 + 2b + 1 \\ 3a^2 + 2b & 2ab - a^2 \end{bmatrix}.$$

3 A general approach to linear boundary value problems

A real linear boundary value problem is the system of equations

$$V'(z) = AV(z), \hspace{1cm} (7a)$$
$$V(-1/2) \in C_-, \hspace{1cm} V(1/2) \in C_+ \hspace{1cm} (7b)$$

5
Figure 1: Two pictures of marginal regions for 2-degrees of freedom systems for the two matrices $M_1$ (left) and $M_2$ (right). Solid lines are the convective marginal region, dashed lines are the overstable marginal region.

where $V(z) : [-1/2, 1/2] \to \mathbb{R}^n$ is a vector-valued function, $A$ is a real $n \times n$ matrix, $C_-, C_+$ are two subspaces of $\mathbb{R}^n$ of dimension $n_-, n_+$ respectively. Our goal is to give conditions on the matrix $A$ and the subspaces $C_{\pm}$ under which non-zero solutions to (7) exist. We will show that the existence of a non-zero solution is granted by the vanishing of analytic expressions depending on the matrix $A$ and the vector spaces $C_{\pm}$.

**Remark 1** In the problem we wish to investigate, the matrix $A$ depends on $a, R, C$, the vector space $C_-$ depends on $\lambda_-, \alpha_-, \beta_-$, and the vector space $C_+$ depends on $\lambda_+, \alpha_+, \beta_+$.

### 3.1 The general case

The general solution to the linear differential equation (7a) is of the form $e^{zA}v$, where $v \in \mathbb{R}^n$. It is hence straightforward that a non-zero solution to (7a) also satisfies (7b) if and only if $e^{A}C_- \cap C_+ \neq \{0\}$.

The fact that $e^{A}C_-$ and $C_+$ have at least a one-dimensional intersection implies that, letting $B$ be a $n \times n_-$ matrix whose columns form a basis of $C_-$, and letting $P$ be a $(n - n_+) \times n$ matrix whose rows are a basis of $C_+$, there exists a non-zero $w \in \mathbb{R}^{n_-}$ such that $Pe^{A}Bw = 0$.

Assume now that $n = 2m$, and that $n_- = n_+ = m$. Then the matrix $Pe^{A}B$ is a $m \times m$ matrix, and the existence of a non-zero $w$ in its kernel is equivalent to the determinant being zero. Hence,

$$\text{det}(Pe^{A}B) = 0$$

is the existence condition of non-zero solutions to (7).

**Remark 2** In our case, the vector spaces $C_{\pm}$ depend linearly on the respective parameters $\lambda_{\pm}, \alpha_{\pm}, \beta_{\pm}$ (each with the corresponding sign), and such parameters appear in only one entry of the matrices $P$ and $B$. It follows that $\text{det}(Pe^{A}B)$ is a function that depends linearly on $\lambda_{\pm}, \alpha_{\pm}, \beta_{\pm}$. The dependence of $\text{det}(Pe^{A}B)$ on $a, R, C, p_3, p_4$ is due to $e^{A}$, and is a composition of algebraic functions with circular or hyperbolic functions.

### 3.2 The parity preserving case

There is a yet more sophisticated method to adopt, that yields much simpler existence conditions. In the hypothesis that $n = 2m$, let $J$ be the diagonal matrix whose entries in the diagonal are an alternating
sequence of +1 and -1, i.e. $J = \text{diag}(1,-1,\ldots,1,-1)$. We then say that

**Definition 3** The boundary value problem is parity-preserving if $\text{JAJ} = -A$ and $JC_+ = C_+$.

The matrix $\text{JAJ}$ has entries in position $i,j$ equal to those of $A$ when $i - j$ is even, and entries in position $i,j$ opposite to those of $A$ when $i - j$ is odd. It follows that $\text{JAJ} = -A$ if and only if $A$ has non-zero entries only in the positions $i,j$ such that $i - j$ is odd. Parity-preserving linear boundary value problems are typical: the system of first order differential equations associated to a differential equation which contains only even derivatives is parity-preserving. It is immediate to prove the following fact.

**Proposition 4** Let $V(z)$ be a solution of a parity-preserving linear boundary value problem. Then the functions $V_\epsilon(z) = (V(z) + JV(-z))/2$, $V_0(z) = (V(z) - JV(-z))/2$ are also solutions of the linear boundary value problem. We call such expressions respectively the even and odd part of $V(z)$.

**Proof** Assume that the vector-valued function $V(z)$ is a solution of equation (7a). Then, necessarily, $V(z) = e^{zA}v$, and hence $V_\epsilon(z) = (e^{zA} + Je^{-zA})v/2$. Computing the $z$-derivative of $V_\epsilon(z)$ one obtains that

$$V_\epsilon'(z) = \frac{1}{2}(Ae^{zA} - JAe^{-zA})v = \frac{1}{2}(Ae^{zA} + AJe^{-zA})v = AV_\epsilon(z).$$

If moreover $V(z)$ satisfies the boundary conditions (7b), that is $V(-1/2) \in C_-$ and $V(1/2) \in C_+$, it follows that

$$V_\epsilon\left(\frac{1}{2}\right) = \frac{1}{2} \left( V\left(\frac{1}{2}\right) + JV\left(\frac{1}{2}\right) \right) \in C_- + JC_+ = C_-,$$

$$V_\epsilon\left(\frac{1}{2}\right) = \frac{1}{2} \left( V\left(\frac{1}{2}\right) + JV\left(\frac{1}{2}\right) \right) \in C_+ + JC_+ = C_+.$$

This, with an identical argument for the odd part, concludes the proof. \qed

Given a vector valued function $V(z)$, the vector valued function $V_\epsilon(z)$ is the one whose even entries are the odd part of the corresponding entries of $V(z)$ and whose odd entries are the even part of the corresponding entries of $V(z)$ (for $V_0(z)$ the same statement is true with the words even and odd exchanged). This observation is commonly used when dealing with systems of differential equations which contain only even derivatives [1]. In such a situation, the space of solutions is the direct sum of even and odd solutions.

From the proposition above, we can conclude that all even (respectively odd) solutions of a parity-preserving boundary value problem are linear combinations of the columns of the matrices $e^{zA} + Je^{-zA}$ (respectively $e^{zA} - Je^{-zA}$), that we call even generators (respectively odd generators). A simple algebraic computation proves that the even generators are simply the odd columns of $e^{zA}$, while the odd generators are the even columns of $e^{zA}$. They both generate a $m$-dimensional space.

Let us denote by $E_1,\ldots,E_m$ the even generators and by $O_1,\ldots,O_m$ the odd generators. As in the previous section, there exists a non-zero even solution $w^iE_i(z)$ (with implicit summation over the index $i$) to the linear boundary value problem if and only if there exists a non-zero $m$-dimensional vector $w$ such that $w^iE_i(1/2) \in C_+$. This is equivalent to the fact that, denoting by $E$ the $n \times m$ matrix whose columns are the vectors $E_i(1/2)$, and denoting by $P$ the $m \times m$ matrix whose rows are a basis of $C^+_m$, there must exist a $w$ such that $PEw = 0$. Taking into account also the odd counterpart, the existence of a non-zero $w$ in the kernel, is equivalent to the scalar conditions

$$\det(PE) = 0, \quad \det(PO) = 0,$$

which are the refined version of the existence conditions found in the previous section. These two equations are typically much simpler functions of the parameters.

**Remark 5** Observe that to compute equations (9) one needs not compute the matrix $e^{zA}$ (which satisfies $e^0 = I_n$), but only determine $m$ independent even (respectively odd) generators of the space of solutions. On the other hand, computation of $e^{zA}$ typically requires the pre and post composition with the $A$-diagonalizing matrix and its inverse, which is not necessary with this approach.
4 The solute Benard problem: stationary convection

Let us apply the arguments above to stationary convection marginal region of the solute Bénard problem. This case corresponds to the choice \( \tau = 0 \) in system (5). In this case, the problem is independent of \( p_\Theta, p_\gamma \).

The general case produces extremely long formulas. For this reason, we devote Section 4.1 to the parity-preserving case, which requires that the boundaries have the same characteristic constants \((\lambda_+ = \lambda_+ = \alpha_+ = \alpha_+ = \beta_+ = \beta_+ = \beta)\). Once computed the equations for the general marginal region, we draw them in some particular cases.

Fixing all the parameters coming from boundary conditions, a point in the effective parameter space \( R_+ \times R^+_R \times R^+_C \) is a convective marginal state when and only when there exist non-zero solutions of the linear system of equations (5) with \( \tau = 0 \) and with boundary conditions (6).

Denoting \( V = (W, DW, D^2 W, D^3 W, \Theta, D\Theta, \Gamma, D\Gamma) \), the \( 8 \times 8 \) matrix associated to the linear system of equations is

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-a^2 & 0 & 2a^2 & 0 & a^2 & 0 & -a^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-R^2 & 0 & 0 & 0 & a^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-C^2 & 0 & 0 & 0 & 0 & 0 & a^2 & 0 \\
\end{bmatrix},
\]

and its characteristic polynomial is

\[
(x^2 - a^2) \left( (x^2 - a^2)^3 + a^2 (R^2 - C^2) \right).
\]

Denoting by \( b \) the only real cubic root of \( a^2 (R^2 - C^2) \), the eigenvalues of the matrix \( A \) are square roots of

\[
a^2, \quad a^2 - b, \quad \left( a^2 + b \right) \pm ib \frac{\sqrt{3}}{2}.
\]

It is hence clear that, excluding the degenerate cases \( b = 0 \), i.e. \( R = C \) (in which case the characteristic polynomial has two quadruple roots equal to \( \pm a \) and the Jordan form of \( A \) has two rank four Jordan blocks), and \( b = a^2 \) (in which case 0 is a double eigenvalue and the canonical form of \( A \) has a rank two Jordan block), the matrix \( A \) has always two real eigenvalues \( \pm a \), four complex eigenvalues \( \pm e_p \pm ie_m \), where

\[
e_p = \frac{\sqrt{2d + b + 2a^2}}{2}, \quad e_m = \frac{\sqrt{2d - b - 2a^2}}{2}, \quad d = \sqrt{a^4 + a^2b + b^2},
\]

and two eigenvalues that are real if \( a^2 - b > 0 \) or purely imaginary if \( a^2 - b < 0 \). Let us denote \( c = \sqrt{a^2 - b} \), with the understanding that \( c \) is either positive real or purely imaginary with positive imaginary part.

4.1 Applications of the parity-preserving formula: marginal equations and marginal regions

The matrix \( A \) is obviously parity preserving. Assuming \( \lambda_+ = \lambda_+ = \lambda, \alpha_+ = \alpha_+ = \alpha \) and \( \beta_+ = \beta_+ = \beta \), also the parity requirement on the boundary conditions is satisfied, and the system is parity-preserving.

The matrix \( P \) introduced in Section 3.1 is

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \beta & 0
\end{bmatrix}
\]
while, a basis of the even solutions introduced in Section 3.2 are

while, a basis of the even solutions introduced in Section 3.2 are

$$E_1 = \begin{bmatrix} 0 & 0 & \cosh(az) & a \sinh(az) \\ 0 & 0 & \cosh(az) & a \sinh(az) \\ 0 & 0 & \cosh(az) & a \sinh(az) \end{bmatrix}, \quad E_2 = \begin{bmatrix} b \cosh(cz) & bc \sinh(cz) & bc^2 \cosh(cz) & bc^3 \sinh(cz) \\ R^2 \cosh(cz) & cR^2 \sinh(cz) & C^2 \cosh(cz) & cC^2 \sinh(cz) \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 2b \cos(z_{e_m}) \cosh(z_{p_e}) & 2b(\epsilon_m \cosh(z_{e_m}) \sinh(z_{p_e}) - \epsilon_m \sinh(z_{e_m}) \cosh(z_{p_e})) \\ b(2a^2 + b) \cos(z_{e_m}) \cosh(z_{p_e}) - \sqrt{3}b \sin(z_{e_m}) \sinh(z_{p_e}) \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 2b \sin(z_{e_m}) \sinh(z_{p_e}) & 2b(\epsilon_m \cos(z_{e_m}) \sinh(z_{p_e}) + \epsilon_e \sin(z_{e_m}) \cosh(z_{p_e})) \\ b(2a^2 + b) \sin(z_{e_m}) \sinh(z_{p_e}) + \sqrt{3}b \cos(z_{e_m}) \cosh(z_{p_e}) \end{bmatrix}$$

The vectors $O_1,...,O_4$ have very similar expressions, which we do not write here. A computation carried over with computer assisted algebra [19] yields a marginal function that is, in the even case, the function

$$C^2 p_{\alpha}(a) q_{\beta,\lambda} (a, R^2 - C^2) - R^2 p_{\beta}(a) q_{\alpha,\lambda} (a, R^2 - C^2),$$

where

$$p_{\alpha} = \cosh \frac{a}{2} + \alpha a \sinh \frac{a}{2}$$

$$q_{\alpha,\lambda} = \cosh \frac{c}{2} \left[ (3b\lambda + d\alpha) \cosh e_p + (3b\lambda - d\alpha) \cos e_m + \right. \left. + (\sqrt{3}e_m + (2b\alpha + 1)e_p) \sinh e_p + (\sqrt{3}e_m - (2b\alpha + 1)e_m) \sin e_m \right] +

+ e \sinh \frac{c}{2} \left[ (b\alpha - 1) \cos e_m + (b\alpha - 1) \cosh e_p + \right. \left. + \alpha(e_m - e_p) \sinh e_p + \alpha(e_m + e_p) \sin e_m \right]$$

In the odd case, the marginal function has the same structure but the functions $p, q$ are

$$p_{\alpha} = \alpha a \cosh \frac{a}{2} + \sinh \frac{a}{2}$$

$$q_{\alpha,\lambda} = \sinh \frac{c}{2} \left[ (3b\lambda + d\alpha) \cosh e_p - (3b\lambda - d\alpha) \cos e_m + \right. \left. + (\sqrt{3}e_m + (2b\alpha + 1)e_p) \sinh e_p - (\sqrt{3}e_m - (2b\alpha + 1)e_m) \sin e_m \right] +

- e \cosh \frac{c}{2} \left[ (b\alpha - 1) \cos e_m - (b\alpha - 1) \cosh e_p + \right. \left. - \alpha(e_m - e_p) \sinh e_p + \alpha(e_m + e_p) \sin e_m \right]$$

(12)
Remark 6 To provide a homogeneous treatment, we do not distinguish the cases in which $c$ is real or purely imaginary. Given the fact that the problem is real-defined, this implies that the functions we obtained can be either real valued or purely imaginary valued. In fact, function (12) is purely imaginary valued in the region $b > a^2$.

4.1.1 Cases in which $\alpha = \beta$

It is easy to observe that, whenever $\alpha = \beta$, the marginal function (11) assumes the form $(C^2 - R^2)p_\alpha(a)p_{\alpha,\alpha}(a, R^2 - C^2)$, and is hence a function of $a$ and $R^2 - C^2$ only, see [2]. This is the direct consequence of the fact that equations (5) can be recast as a system of equations for the fields $W, \Phi = \Gamma - \Theta, \Psi = \Gamma + \Theta$

$$\left\{ \begin{array}{l}
(D^2 - a^2) W - a^2 \Phi = 0 \\
(D^2 - a^2) \Phi + (C^2 - R^2) W = 0,
\end{array} \right.$$

When $\alpha = \beta$, also the boundary conditions become functions of $W, \Phi, \Psi$. Solutions of the system can be found solving its first two equations. Hence, existence conditions depend on $a, R^2 - C^2$ only, and are those of the simple Bénard problem [1]. The extreme cases are listed below.

A. Rigid boundaries ($\lambda = 0$) and fixed temperature and concentration ($\alpha = \beta = 0$): the marginal equations for the even and odd case are respectively

$$c \tanh \frac{c}{2} = \frac{(\sqrt{3}c_p - e_m) \sin e_m + (\sqrt{3}c_m + e_p) \sinh e_p}{\cos e_m + \cosh e_p},$$

$$c \coth \frac{c}{2} = \frac{(\sqrt{3}c_p - e_m) \sin e_m - (\sqrt{3}c_m + e_p) \sinh e_p}{\cos e_m - \cosh e_p}.$$

B. Stress-free boundaries ($\lambda = \infty$) and fixed temperature and concentration ($\alpha = \beta = 0$): the marginal equations for the even and odd case are respectively $\cosh[c/2] = 0$ and $\sinh[c/2] = 0$.

C. Rigid boundaries ($\lambda = 0$) and fixed temperature and concentration flows ($\alpha = \beta = \infty$): the marginal equations for the even and odd case are respectively

$$d \frac{c}{\cot h} \frac{c}{2} = \frac{(e_m + \sqrt{3}c_p) \sin e_m + (\sqrt{3}c_m - e_p) \sinh e_p}{\cos e_m - \cosh e_p},$$

$$d \frac{c}{\tanh} \frac{c}{2} = \frac{(e_m + \sqrt{3}c_p) \sin e_m + (\sqrt{3}c_m - e_p) \sinh e_p}{\cos e_m - \cosh e_p}.$$

D. Stress-free boundaries ($\lambda = \infty$) and fixed temperature and concentration flows ($\alpha = \beta = \infty$): the marginal equations for the even and odd case are respectively

$$\frac{c}{2} \tanh \frac{c}{2} = \frac{e_m \sin e_m - e_p \sinh e_p}{\cos e_m + \cosh e_p},$$

$$\frac{c}{2} \coth \frac{c}{2} = \frac{e_m \sin e_m + e_p \sinh e_p}{\cos e_m - \cosh e_p}.$$

4.1.2 Cases in which $\alpha \neq \beta$

The remaining relevant cases are those in which the boundary conditions on temperature and concentration are extremal but different. A quick overview of (11) shows that the exchange of $\alpha$ with $\beta$ in the boundary conditions yields an exchange of $R^2$ with $C^2$ in the equation. We hence write only two cases, with the understanding that the other two can be obtained by substituting $R^2/C^2$ with $C^2/R^2$. 

10
E. Rigid boundaries ($\lambda = 0$), fixed heat flow and fixed concentration ($\alpha = \infty$, $\beta = 0$)

$$\frac{R^2}{C^2} \left[ c \tanh \frac{c}{2} \left[ (e_m + \sqrt{3}e_p) \sin e_m + (\sqrt{3}e_m - e_p) \sinh e_p \right] - d(\cos e_m - \cosh e_p) \right] =$$

$$a \tanh \frac{a}{2} \left[ (\sqrt{3}e_p - e_m) \sin e_m + (\sqrt{3}e_m + e_p) \sinh e_p - c \tanh \frac{c}{2} (\cos e_m + \cosh e_p) \right],$$

$$\frac{R^2}{C^2} \left[ c \coth \frac{c}{2} \left[ (\sqrt{3}e_m - e_p) \sinh e_p - (e_m + \sqrt{3}e_p) \sin e_m \right] + d(\cos e_m + \cosh e_p) \right] =$$

$$a \coth \frac{a}{2} \left[ (e_m - \sqrt{3}e_p) \sin e_m + (\sqrt{3}e_m + e_p) \sinh e_p + c \coth \frac{c}{2} (\cos e_m - \cosh e_p) \right].$$

F. Stress-free boundaries ($\lambda = \infty$), fixed heat flow and fixed concentration ($\alpha = \infty$, $\beta = 0$)

$$3a \frac{R^2}{C^2} \tanh \frac{a}{2} = c \tanh \frac{c}{2} - 2 \frac{e_m \sin e_m - e_p \sinh e_p}{\cos e_m + \cosh e_p},$$

$$3a \frac{R^2}{C^2} \coth \frac{a}{2} = c \coth \frac{c}{2} - 2 \frac{e_m \sin e_m + e_p \sinh e_p}{\cos e_m - \cosh e_p}.$$

4.1.3 Plots of significant parity-preserving cases

The formulas above can be used to draw plots in $a, R$-space with fixed $C$. We use these plots to illustrate graphically some properties of the marginal regions. Figure 2 represents the typical structure of the marginal region when $\alpha \neq \infty$. The plot shows an alternation of non-intersecting curves associated to even and odd solutions.

Figure 2: The marginal region associated to existence of even solutions (continuous lines) and odd solutions (dashed lines) with rigid, fixed temperature, and fixed solute boundary conditions, and with $C = 20$.

Figure 3 shows that in the case $\alpha = \beta$ the marginal function depends on $a$ and $R^2 - C^2$ while, in the case $\alpha \neq \beta$, such property fails. In particular, only in the first case the marginal region in the $a, R^2$ space
is vertically translated by changes of $C$ and hence the $a$ coordinate of the minimum is independent of $C$. The left frame corresponds to rigid boundaries, and fixed temperatures and solute concentrations, the right frame corresponds to rigid boundaries, fixed temperatures, and fixed solute fluxes. In both frames only the lowest branch, corresponding to even solutions, is shown for $C = 0$ and $C = 30$.

![Figure 3: Lower branch of the marginal region in the space $a, R^2$ for $C = 0$ (continuous) and $C = 30$ (dashed). On the left, a case in which $\alpha = \beta$ shows the dependence on $R^2 - C^2$, on the right, a case in which $\alpha \neq \beta$.](image)

When the boundary condition is that of fixed heat flux, the marginal region significantly changes. In fact, the lowest branch of the marginal region approaches the $a=0$ plane at a finite value. Details of this case are given in Section 6.

For every boundary condition on velocity and solute except that of fixed solute flow, such limit turns out to be independent of $C$. We show this fact in Figure 4. We finally show, in Figure 5, that when the boundary conditions are of fixed heat flux and fixed solute flux, the asymptotic behavior is again dependent on $C$.

5 The solute Bénard problem: overstability

Referring once again to the linear system of equations (5) with boundary conditions (6), the overstable marginal region is the set of points in the effective parameter space $\mathbb{R}^+_a \times \mathbb{R}^+_R \times \mathbb{R}^+_C$ for which there exist non-zero solutions of the linear system of equations $DV = AV$ with $V$ as in Section 4, and

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-a^2(a^2 + i\tau) & 0 & 2a^2 + i\tau & 0 & a^2 & 0 & -a^2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-R^2 & 0 & 0 & 0 & a^2 + i\tau p_\phi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-C^2 & 0 & 0 & 0 & 0 & 0 & a^2 + i\tau p_\gamma & 0 \\
\end{bmatrix},$$

The characteristic polynomial of $A$ is a fourth-order polynomial in $x^2$ and hence its zeroes can be expressed using Ferrari’s formulas. Such expressions are cumbersome to write and difficult to manage,
The asymptotic for \( a \to 0 \) of the marginal regions for fixed heat flow, fixed concentration, and \( C = 10 \) (continuous) or \( C = 25 \) (dashed). The left frame corresponds to rigid boundary conditions while the right frame corresponds to stress-free boundary conditions.

Figure 4: The asymptotic for \( a \to 0 \) of the marginal regions for fixed heat flow, fixed concentration, and \( C = 10 \) (continuous) or \( C = 25 \) (dashed). The left frame corresponds to rigid boundary conditions while the right frame corresponds to stress-free boundary conditions.

even with an algebraic manipulation software. Nonetheless, formula \( \det(\text{Pe} A B) \) yields a complex-valued function \( f_\tau(a, R, C; p_\theta, p_\gamma, \alpha^\pm, \beta^\pm, \lambda^\pm) \) which depends linearly on the parameters \( \alpha^\pm, \beta^\pm, \lambda^\pm \) and through composition of algebraic and exponential functions on the parameters \( a, R, C, p_\theta, p_\gamma, \tau \). The zero level set of \( f_\tau \) is a codimension 2 manifold for every fixed \( \tau \). Varying \( \tau \) in the positive-real numbers, this codimension 2 manifold spans a hypersurface, which is the overstable marginal region.

In the particular case of \( p_\theta = p_\gamma = 0 \), the characteristic polynomial has eigenvalues that can be expressed as zeroes of a cubic polynomial in \( x^2 \) and a quadratic polynomial. The fact that, in this case, the algebraic expression of the eigenvalues are Cardano formulas, does not reduce much the complexity of the expressions.

Other than granting the existence of such functions, this approach can be used to compute numerically the overstable marginal region. For every choice of the parameters, the matrix \( A \) is a numerical matrix, and the Mathematica’s function \texttt{MatrixExp} [19] allows to define two functions \( \text{Re}(f_\tau(a, R)), \text{Im}(f_\tau(a, R)) \) which, for every choice of \( \tau \), have common zeroes that belong to the overstable marginal region. A Newton method (Mathematica’s \texttt{FindRoot}) allows to numerically determine such zeroes, and gives the curves in Figure 6.

As anticipated in Section 2.1, the main feature of this figures is that overstable marginal region is a curve whose end-points lie in the convective marginal region.

6 The solute Bénard problem: investigation around \( a = 0 \)

One of the striking features of marginal regions is that, under some boundary conditions (see Figures 4, 5, 6), they have a finite asymptotic when \( a \) tends to zero. This phenomenon has already been described in [14, 15, 16, 17]. In this section we use the analytic expression obtained with the approach for non-symmetric boundaries to show that.

\textbf{Proposition 7} The marginal region has a finite asymptotic as the wave number \( a \) tends to zero if and only if both boundary conditions are of fixed heat flux. In which case, the convective marginal region tends to the \( a=0 \) plane.

\textbf{Proof} The overstable marginal region cannot approach the \( a=0 \) plane. This can be proved by substituting in system (5) the expansion in \( a \) of the fields \( W, \Theta, \Gamma \) and of the parameter \( R \), showing that the
zero-order term in $a$ of the equations does admit only the zero solution, and that the same is true for all orders (see [18, 17] for the technique). Exceptional is the case $p_0 = 0$ or $p_\gamma = 0$, which we will not discuss here.

Hence, only the convective marginal region can possibly approach the $a = 0$ plane. We therefore compute the Taylor expansion around $a = 0$ of the convective marginal function given by (8) with the matrix $A$ defined in (10). A computation shows that the powers of the matrix $A$ up to five have terms that are not divisible by $a$, while the power $A^6$ is divisible by $a^2$. It follows that $a^2$ divides $A^k$ for $k \geq 6$. Therefore, the coefficient of $a^0$ in a Taylor expansion of the marginal function can be computed using the truncation of $e^A$ to order five, which means that it suffice to compute

$$\det \left( P \left( I_8 + A + \frac{A^2}{2} + \cdots + \frac{A^5}{5!} \right) B \right).$$

With a slight change in the convention on matrices $P, B$, that is writing boundary conditions (6) as convex combinations

$$(1 - \lambda_\pm)DW \pm \lambda_\pm D^2W = (1 - \alpha_\pm)\Theta \pm \alpha_\pm D\Theta = (1 - \beta_\pm)\Gamma \pm \beta_\pm D\Gamma = 0,$$

the coefficient of $a^0$ turns out to be

$$(\alpha_+ - \alpha_-)(\beta_- - \beta_+)(1 + 3\lambda_+ + 3\lambda_- 5\lambda_+ \lambda_-)/12,$$

and it is obviously zero only when $\alpha_+ = \alpha_- = 1$ or $\beta_+ = \beta_- = 1$ (that correspond to fixed heat flux or fixed solute flow). This means that in all other cases, the marginal region will not approach the $R$ axis.

Assuming to be in the case of fixed heat flux, one has to investigate the coefficient of $a^2$ of the Taylor expansion. Such term can be computed using the truncation of the exponential $e^A$ up to order eleven, and it turns out to be

$$\frac{\beta_- - \beta_+ - 1}{8640} \left( R^2 - 720 + 8 \left( R^2 - 270 \right) (\lambda_- + \lambda_+) + 5 \left( 11R^2 - 720 \right) \lambda_- \lambda_+ \right).$$
The expression above can be zero if

\[ R^2 = \frac{720}{1 + 8(\lambda_+ + \lambda_-) + 55\lambda_+\lambda_-} \]  
which is, surprisingly, independent from \( C \). So the solute does not have a stabilizing effect for any value of the gradient \( C^2 \). This independence was shown numerically in [15, 14]. The other possibility is that \( \beta_+ = \beta_- = 1 \). Assuming this, equating to zero the coefficient of \( a^4 \) one obtains an asymptotic limit which has the same expression of (13) with \( R^2 \) replaced by \( R^2 - C^2 \). The dependence of the limit on \( R^2 - C^2 \) is expected, as observed at the beginning of Section 4.1.1.

In the last case yet to analyze \( (\beta_+ = \beta_- = 1, \text{but one of the } \alpha \neq 1) \), the marginal region does not approach the \( a \) axis. In fact, the coefficient of \( a^2 \) is never zero. This fact would completely change if the fluid was salted form above.

With all the choices of boundary parameters, Prandtl number, and Schmidt number we have made, the overstable marginal region, if it exists, lies below the convective marginal region, and it connects to it at two positive values of wave number \( a \) (see Figure 6). We observed that, increasing the ratio \( p_\gamma / p_\theta \), the overstable marginal region lowers, and that the \( a \)-value of the first intersection point moves closer and closer to the \( a = 0 \) plane. It is reasonable that, when \( p_\theta = 0 \) convective and overstable marginal region both converge to the same value of \( R \) as \( a \) tends to zero.

7 Acknowledgements

This work was partially supported by GNFM-INdAM project “Progetto giovani ricercatori 2010: Modelli classici e quantistici in termodinamica estesa e problemi di stabilità in fluidodinamica”.

---

Figure 6: Two plots of convective and overstable marginal regions. Superposed to the overstable marginal region, bars proportional to the imaginary part \( \tau \) of the eigenvalue. The left plot refers to rigid, fixed temperature and fixed solute concentration boundary conditions, with \( C = \sqrt{5000}, p_\theta = 0.1, p_\gamma = 0.4 \). The right plot refers to rigid, fixed heat flux and fixed solute concentration boundary conditions, with \( C = 24, p_\theta = 0.1, p_\gamma = 8 \).
References


