The nuclear models for pedestrian
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V.R. Manfredi ¹

**Introduction**

For ten years we gave a course in Nuclear Spectroscopy to students attending the Science Faculty at Padua University and also to Erasmus students.

At the request of the students we collected the lessons in the present article.

We are grateful to Physics Italian Society for the editorial aid.

For many years the theories of atomic nucleus were essentially based on nuclear models (macroscopic and/or microscopic) [Ref 1,2,3,4]. Only recently a complete “abinitio” theory has been developed, including relativistic effects and mesons degree of freedom [Ref 5].

The atomic nucleus \(^{n}X_{z}^{n}\) is mainly based on nucleons in interaction, \(Z\) protons and \(N\) neutrons, with \(A=Z+N\).

The main idea of theoretical nuclear physics is that the description of all properties of atomic nuclei (the binding energy of the ground state, the spectrum of excitation, the transition probabilities, the electric and magnetic moments, …) are obtained by the interaction of free nucleons.

The complexity of the nucleon-nucleon interaction and the difficulties in solving Schrödinger equation for many body problem (\(A\geq10\)) lead to the introduction of a lot of phenomenological nuclear models each one stressed on one aspect of nuclear properties.

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In nuclear reactions and nuclear spectroscopy fields, two different theories have been developed.

The first one is models as independent particles, where each nucleon is in the average mean field and the interaction between other nucleons is weak. In the nuclear reaction field the compound nucleus model was proposed, otherwise in the nuclear spectroscopy field the shell one.

1. The compound nucleus

Following N. B. [Ref 6], the so-called compound nucleus model of nuclear reactions is briefly discussed.

It is assumed that when some target nucleus A is bombarded by an incident nuclear particle $a$, both may coalesce to form a compound nucleus ($A+a$).

In the compound nucleus ($A+a$), there are assumed to be strong interactions among all the nucleus. The incident nuclear particle $a$ loses its independent identity, and the total energy of the excited compound nucleus is shared in a complicated manner by all nucleons involved.

The compound nucleus ($A+a$) is taught of as being in a quasi-stationary quantum state, whose mean life is long ($\sim 10^{-16} \text{ sec}$) compared with the time used by a proton to cross nucleus ($\sim 10^{-22} \text{ sec}$).

Identically in the same level of excitation the same compound nucleus can be produced by the collision (usually at a different bombarding energy) of other nuclei, say, B and b, so that it is possible to have
\( (A+a) = [B+b] \).

In the N. Bohr postulates, the properties of the compound nucleus \((A+a)\) are independent of its mode of formation, i.e. the compound nucleus “forgets” how it was formed.

The second step in the nuclear reaction is the dissociation of the compound nucleus. This dissociation can generally take place in a large number of ways, sometimes called “exit channels,” subject to the conservation laws for mass-energy, charge angular momentum, etc.

The competition among various alternative modes of disassociation does not depend on the manner in which the compound nucleus was formed, i.e., on the “entrance channel”. Schematically, any nuclear reaction in which a compound nucleus is formed can be represented as

<table>
<thead>
<tr>
<th>Entrance channel</th>
<th>Compound nucleus</th>
<th>Exit channels</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A+a)</td>
<td>(elastic scattering)</td>
<td></td>
</tr>
<tr>
<td>(A^*+a)</td>
<td>(inelastic scattering)</td>
<td></td>
</tr>
<tr>
<td>(B+b)</td>
<td>(nuclear transformation)</td>
<td></td>
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<td>(C+b+c)</td>
<td>(nuclear transformation)</td>
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<tr>
<td>(D+d)</td>
<td>(nuclear transformation)</td>
<td></td>
</tr>
<tr>
<td>Etc.</td>
<td>(nuclear transformation)</td>
<td></td>
</tr>
</tbody>
</table>
The asterisk, as in \( A^\ast \), denotes an excited level of a nucleus. As an explicit example, the reaction on \( F^{19} \) in which a proton is captured and an \( \alpha \) particle is emitted would be written

\[ _9F^{19} + _1H^1 \rightarrow (_{10}Ne^{20}) \rightarrow _2He^4 + _4O^{16} \]

The same compound nucleus might instead emit a neutron, leaving \( _{10}Ne^{19} \) as the residual nucleus, according to

\[ _9F^{19} + _1H^1 \rightarrow (_{10}Ne^{20}) \rightarrow _0n^1 + _{10}Ne^{19} \]

In the more compact notation which is usually used for nuclear reactions, these two competing reactions would be written \( F^{19}(p, \alpha) \) and \( F^{19}(p, n)Ne^{19} \) without explicit designation of the compound nucleus.

![Figure 1: Bohr’s model of a neutron-nucleus collision](image)

2. The residual interaction

In order to take into account also the residual interaction between the nucleus, the total nuclear Hamiltonian \( H \) may be written:

\[ (R.I.I)H = H_0 + V_R, \]
where

\[(R.I.2) H_0^i = \sum_i H_0^{(i)} = \sum_i [T_i + V_i] = \sum_i \frac{p_i^2}{2m_i} + V_i \]

and

\[(R.I.3) V_i = \sum_i V_{i0} - \sum_i V_{i1} \]

$H_0^{(i)}$ is the unperturbed single particle and $V_R$ is the nucleon-nucleon residual interaction.

The residual interaction $V_R$ may be obtained by self consistent Hatree – Fock equation, starting from the free nucleon-nucleon interaction.

Very often a phenomenological approach was followed, extract the single- particle energies from closed shell nuclei plus one particle ($^{17}O$, $^{41}Ca$, $^{209}Pb$, $^{209}Bi$, (see fig.))

But in order to determine the single particle spectrum $\varepsilon_i$ and the wave functions $u_i$, the Schrödinger equation

\[(R.I.4) H_0^{(i)} u_i = \varepsilon_i u_i \]

is solved.

The antisymmetric wave function $Q_\alpha$, in order to take into account the Pauli principle, is the solution of the equation:

$H_0 Q_\alpha = E_\alpha^{(0)} Q_\alpha$,

where

\[(R.I.5) E_\alpha^{(0)} = \sum_i \varepsilon_i \]
In order to obtained the eigenstates of $H$, we diagonalize $H$ on the basis of all configurations $\phi_n$. The Schrödinger equation is:

\[(\text{R.I. 6}) \quad H \phi_n = E_n \psi,\]

where

\[(\text{R.I. 7}) \quad \psi_n = \sum_a C_{n\alpha} \phi_\alpha.\]

The eigenvalues $E_n$ and the eigenvectors $C_n$ are the solutions of the secular problem:

\[(\text{R.I. 8}) \quad \sum_\beta E^{(0)}_{\alpha\beta} \delta_{\alpha\beta} + \langle \phi_\alpha | V_R | \phi_\beta \rangle C_{n\beta} = E_n C_{n\alpha},\]

where $\langle \phi_\alpha | V_R | \phi_\beta \rangle$ are the matrix elements of the operator $V_R$, function of the two body matrix elements $V_\delta$ (Ref.).

In order to avoid the difficulty of infinite number of terms in (R.I.8), a truncation of model space/see can be used and we take into account just “active orbits” (see fig.).

With the truncation of the model space, the structure of (R.I. 8) is:

\[(\text{R.I. 9}) \quad \left( \begin{array}{c}
(E_1^{(0)} - E_n) + \langle \phi_1 | V_R | \phi_1 \rangle \langle \phi_1 | V_R | \phi_2 \rangle \cdots \langle \phi_1 | V_R | \phi_M \rangle \\
(E_2^{(0)} - E_n) + \langle \phi_2 | V_R | \phi_2 \rangle \cdots \\
\vdots \\
(E_M^{(0)} - E_n) + \langle \phi_M | V_R | \phi_M \rangle \\
\end{array} \right) \]

The energies $E_n$ are the theoretical spectrum of the system to compare with the experimental data. The eigenvectors $C_n$ are determined by the condition

\[(\text{R.I. 10}) \quad \sum_\alpha C_{n\alpha} = \delta_{nm}.\]
By the knowledge of the wave function \( \psi_n \) is possible to calculate all observable quantities such as the quadruple moment \( Q_1 \), the magnetic moment, \( \mu_0 \) ...

As an example, we report in (R.I.11) the quadruple moment \( Q \) of the lever \( n \):

\[
(R.I.11) \; Q \sim \langle \psi_n | \hat{Q} | \psi_n \rangle = \sum_{\alpha - \beta} C_{\alpha \alpha'} C_{\alpha' \beta} \langle \varphi_{\alpha} | \hat{Q} | \varphi_{\beta} \rangle,
\]

where the matrix elements \( \langle \varphi_{\alpha} | \hat{Q} | \varphi_{\beta} \rangle \) are functions of the structure of the operator \( \hat{Q} \) and of basis state \( \varphi_i \).

3. Rotational model

If there are many nucleons (neutrons and protons) outside the closed shells, there is a permanent deformation of the atomic nuclei. This deformation is in fact, the competition of the closed shells (spherical nuclei) and the nucleons outside the closed shells. In zero order approximation, it is possible to disentangle the motion of nucleons by the collective (adiabatic approximation).
The validity of the adiabatic approximation is confirmed experimentally by the existence of rotational motions. If the deformation is axially symmetric, the energies of the rotational model are:

\[ E_{I,K} = \frac{\hbar^2}{2I} \left[ I(I+1) - K^2 \right] + \frac{\hbar^2}{2I_3} K^2, \]

where \( I \) is the total angular momentum of the nucleus and \( K \) is the projection of \( I \) on the symmetric axis \( Z \).

\( J = J_1 = J_2 \) and \( J_3 \) are the moments of inertia, referred to principal axes. In fig the spectrum of \(^{168}\text{Er}\) is reported for different values of \( K \). The so-called “ground state band” correspond to \( K = 0 \) and the (RM1) becomes:

\[ E_I = \frac{\hbar^2}{2J} I(I+1). \]

In the adiabatic approximation the moment of inertia \( J \) is constant in the band and it is very easy to calculate the ratio between the energies of the different levels:

\[ \frac{E_4}{E_2} = \frac{10}{3}, \frac{E_6}{E_2} = 7, \frac{E_8}{E_2} = 12, \ldots. \]

The adiabatic approximation allows one that the nucleus wave function may be factorized as:

\[ \psi = \Phi_{\text{vib}} \times D_{\text{rot}} \times \chi, \]

where

\( \Phi_{\text{vib}} \) describes the vibrations of the nucleus around the spherical scape,

\( D_{\text{rot}} \) describes collective rotational motion,

\( \chi \) describes the motion of the nucleons in the deformed field.
In the rotational motion we are only interested to $\chi$ and $D$.

The basic hypotheses are:

*) Nuclei with non spherical shape.

*) Factorization of the nuclear wave function.

We call OXYZ (S) is a fixed system (a lab system) and $Ox'y'z'(s')$ a system in the nucleus.

The orientation of $s'$ is determined by three Euler angles $\theta_i(\alpha, \beta, \gamma)$

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$k^{\pi} = 3^-$

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$k^{\pi} = 4^-$

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$k^{\pi} = 2^+$

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$^{168}_{68}$Er

$^{168}_{68}$Er$_{100}$

$^{168}_{68}$Er

Figure 3: Spectrum of $^{168}$Er

3.1. Hamiltonian of the rotational model

The Hamiltonian of the rotational model may be written in the form:
(RM5) \[ H_T = T_{\text{rot}} + H_{\text{int}}(x') + H_{\text{compl}}, \]

where

\( T_{\text{rot}} \) is the kinetic energy of rotation,

\( H_{\text{int}}(X') \) is the Hamiltonian of intrinsic motion,

\( H_{\text{compl}} \) is the Hamilton of the coupling between the intrinsic motion and the rotational motion.

The adiabatic approximation allows one to write the function \( \Psi \) as:

\[ \Psi = \chi(E') D(\Theta) \]

If \( \vec{R} \) is the angular momentum of the collective motion, \( \vec{J} \) is the angular momentum of the intrinsic motion and \( I \) is the total angular of the nucleus:

(RM6) \[ \vec{I} = \vec{R} + \vec{J} \]

for even-even nuclei (Z even N even), \( \vec{J} = 0 \) and (RM6) \( \vec{I} = \vec{R} \),

the analysis of the experimental spectra suggests two different hypothesis:

*) the nuclear deformation is axially symmetric.

*) The nuclear deformation has a symmetric plane.

If the rotational motion is similar to a rigid body:

\[ T_{\text{rot}} = \frac{\hbar^2}{2} \left( \frac{R_1^2}{J_1} + \frac{R_2^2}{J_2} + \frac{R_3^2}{J_3} \right) 1 \rightarrow x' \ 2 \rightarrow y' \ 3 \rightarrow z', \]

where \( R_1 \), \( R_2 \) and \( R_3 \) are the components of the angular momentum of the nucleus.

If \( OZ' \) is the symmetry axes, we have:

(RM7) \[ J_1 = J_2 = J, \]
the nuclear Hamiltonian may be written:

\[ H = T_{rot} + \frac{\hbar^2}{2J} (\vec{R}^2 - R_z^2) + \frac{\hbar^2}{2J^3} R_z^2, \]

where

\[ \vec{R}^2 = R_1^2 + R_2^2 + R_3^2 = R_x^2 + R_y^2 + R_z^2. \]

For even-even nuclei the rotational Hamiltonian may be written:

\[ (RM7) \quad H = \frac{\hbar^2}{2I} (I^2 - I^3) + \frac{\hbar^2}{2J^3} I_z^2, \]

where \( \vec{I} \) is the total angular momentum and \( I_z \) is the component along the \( Z' \) axe.

The energy spectrum for the rotational motion is:

\[ (RM8) \quad E_{I,K} = \frac{\hbar^2}{2J} [I(I+1) - K^2] + \frac{\hbar^2}{2J^3} K^2, \]

4. Wave Functions

The eigenfunctions of the rotational Hamiltonian are:

\[ D_{MK}^I(\theta) \]

\[ I^2 D_{MK}^I = I(I+1) D_{MK}^I \]

\[ (RM9) \quad I_z D_{MK}^I = M_{MK}^I \]

\[ I_z D_{MK}^I = K D_{MK}^I. \]

Where \( M \) is the projection of the total angular momentum \( \vec{I} \) on the \( 0Z \) axe and \( K \) is the projection on the \( OZ' \) axe.
The normalized wave functions are:

\[ |IMK> = \frac{2I+1}{8\pi^2} D_{MK}^I(\theta) \chi_0 \]

4.1 First symmetry property of the wave functions

*) Symmetry property around the axis \( OZ' \)

The wave function is invariant for the rotation of the Euler angles \( (\alpha, \beta, \gamma + \Phi) \) then:

\[ \Psi_{MK}^I(\alpha, \beta, \gamma + \Phi) = \Psi_{MK}^I(\alpha, \beta, \gamma) \]

\( \chi_0 \) is invariant for the rotation and

\[ D_{MK}^I(\alpha, \beta, \gamma + \Phi) = e^{i\Phi} D_{MK}^I(\alpha, \beta, \gamma) \]

As the consequence of the symmetry properties \( K=0 \) and \( \mathcal{J} \) is perpendicular to the \( OZ' \) axes.

Figure 4: Projection of \( I \) on \( Z \) axis and \( Z' \) axis
4.2. second symmetry property

This kind of symmetry is suggested by even values of \( I = (0 \rightarrow 2 \rightarrow 4 \ldots) \) for the transformation

\[(\alpha, \beta, \gamma) \rightarrow (\alpha, \beta+\pi, -\gamma),\]

we have:

\[(RM13) (\alpha, \beta, \gamma) \rightarrow (\alpha, \beta+\pi, -\gamma)\]

\[(RM14) R_{y}(\pi) D_{MK}^{i}(\theta) = D_{MK}^{i}(\alpha, \beta, \pi - \gamma) = (-1)^{l-K} D_{MK}^{i}(\alpha, \beta, \gamma),\]

for \( K=0 \)

\[D_{MK}^{i}(\alpha, \beta, \pi, -\gamma) = (-1)^{l-K} D_{MK}^{i}(\alpha, \beta, \gamma).\]

As a consequence of symmetry properties of the transformation \((\alpha, \beta, \gamma) \rightarrow (\alpha, \beta+\pi, -\gamma)\) \( I \) must be even.

The wave function is:

\[(RM15) |IMO\rangle = \sqrt{\frac{2l+1}{8\pi^2}} D_{MO}^{l}(\alpha, \beta, \gamma) \chi_0,\]

with \( I=0,2,4..., \)

in term of spherical harmonic we have:

\[(RM16) |IMO\rangle = \frac{1}{\sqrt{2\pi}} Y_{M}^{l}(\beta, \alpha) \chi_0.\]

A rotational band is a set of states with the same intrinsic function, but different values of \( I \).

5. The problem of the moment of inertia

In order to calculate the moment of inertia a detailed motion of nucleons must be calculated.
Historically, two different approaches were followed. The first one was similar to a rigid body \( (J_{\text{rig}}) \), the second one was similar to an irrotational liquid \( (J_{\text{irrot}}) \).

In Fig we compare \( J_{\text{rig}}, J_{\text{irrot}} \) and the so called “\( J_{\text{sp}} \)” (taken from \( 2^+ \) state). The agreement is not good at all.

In 1961 S.G. Nilsson and O. Prior obtained a good agreement with the “experimental” data, by the introduction of a pairing interaction \(^2\) between the nucleons.

The result of Nilsson and O.Prior is:

\[
J = 2\hbar^2 \sum_{V} \frac{\langle V | J_1 | V \rangle^2}{E_{\nu} + E_{\nu}'} \left( U_{\nu} V_{\nu} - V_{\nu} U_{\nu} \right)^2,
\]

where \( E_{\nu} = \sqrt{\left( \sum_{\nu} - x \right)^2 + \Delta^2}, \quad U_{\nu}^2 + V_{\nu}^2 = 1 \)

---

\(^2\) the matrix element of the pairing interaction is \( B_{j_1,j_2} | V_{p1} j_3 j_4 X = -G \delta_{j_1,j_1} \delta_{j_3,j_3}, \quad G > 0 \)
6. Electromagnetic transitions in Rotational Model

The transition probability $T$ between an initial state $\Psi(I_i, M_i)$ and a final state $\Psi(J_f, M_f)$ is given by:

$$(ET1) T = \frac{1}{\tau} = \frac{\Gamma}{\hbar},$$

Where $\Gamma$ is the width of the nuclear level $\Psi$.

For electromagnetic transitions, the selection rule is:

$$|I_i - I_f| \leq \lambda \leq I_i + I_f,$$

where $\lambda$ is multipolarity of the emitted transition.

The consequence of the parity conservation is:

- $$(ET2) \pi_i \pi_f = (-1)^\lambda$$ for electric transitions
- $$(\pi_i \pi_f = (-1)^{I_i + 1}$$ for magnetic transitions.

The structure of the multiple electric operator $m_e(\lambda \rightarrow \mu)$ is:

$$(ET3) m_e(\lambda, \mu) = \sum_{i=1}^{\lambda} r_i J_\mu(\vartheta, \varphi_i)$$

The transition probability $T(\lambda)$ is given by:

$$(ET4) T(\lambda) = \frac{8\pi(\lambda+1)}{3(2\lambda+1)!} \frac{1}{\hbar c} \left(\frac{\Delta E}{\hbar c}\right)^{2\lambda+1} \beta(\lambda),$$

where $\Delta E$ is the difference between the energy of the final and the initial state and $\beta(\lambda)$ is the reduced transition probability.

$$(ET5) \beta(\lambda, I_i \rightarrow I_f) = \sum_{M_f} \langle J_f M_f | M(\lambda, \mu) | J_i M_i \rangle^2$$
By the use of the Winger-Eckert theorem and the properties of C.G. coefficients, we have:

\[ (ET6) \langle I_f, M_f | m(\lambda, \mu) | I_i, M_i \rangle = (I_i, \lambda M_i, \mu | I_f, M_f) \frac{\langle I_f | m(\lambda) | I_i \rangle}{\sqrt{2I_f + 1}} \]

\[ (ET7) \sum_{M_{i \mu}} (I_i, \lambda M_i, \mu | I_f, M_f) = \frac{1}{2I_f + 1} \sum_{M_{i \mu}} (I_i, \lambda M_i, \mu | I_f, M_f)^2 \]

\[ (ET8) \beta(\lambda, I_i \rightarrow I_f) = \sum_{I_f} (I_i, \lambda M_i, \mu | I_f, M_f)^2 \frac{\langle I_f | m(\lambda) | I_i \rangle^2}{2I_f - 1} = \frac{2I_f + 1}{2I_f + 1} \frac{\langle I_f | m(\lambda) | I_i \rangle^2}{2I_f - 1} \]

\[ = \frac{\langle I_f | m(\lambda) | I_i \rangle^2}{2I_f + 1} . \]

Very often the reduced transition probabilities \( B(\lambda) \) are expressed as a function of a single particle transition. For the transition \( I_i = 2 \rightarrow I_f = 0 \) the single value is (Weisskopf unit):

\[ (ET9) B(E2)_{x,p} = \frac{1}{4\pi} \left[ \frac{3}{5} R_0^2 \right]^2 = G \Lambda^{4/3} \times 10^{-2} \text{ cm}^4 . \]

7. Electric Quadruple Moments

The electric quadruple moment is given by:

\[ (EQ1) \mathcal{Q} = \langle I, M = I | Q_0^{(2)} | I, M = I \rangle, \]

where

\[ (EQ2) Q_0^{(2)} = \sum_{i=1}^{Z} (\zeta Z_i^2 - r_i^2) = \sqrt{\frac{16\pi}{5}} \sum_{i=1}^{Z} r_i^2 Y_0^2(\theta_i, \varphi_i) \]

\( Q_0^{(2)} \) is a particular case \( (\lambda=2, \mu=0) \) of the electric multiple operator \( m_\epsilon(\lambda \rightarrow \mu) \) given by:

\[ (EQ3) m_\epsilon(\lambda, \mu) = \sum_{i=1}^{Z} r_i^{(\lambda)}(\theta_i, \varphi_i) . \]
In fact

\[ (EQ4) m_{\nu}(2,\mu) = \sqrt{\frac{5}{16\pi}} e^{Q^{(2)}_{\nu}}, \]

where \( Q^{(2)}_{\nu} \) is the electric quadrupole moment in the lab system.

In the system \( S' \), taken into account is the relation:

\[ (EQ5) Y_{\nu}^{\lambda}(\theta,\phi) = \sum_{\mu} D_{\mu\nu}^{\lambda}(\theta,\phi) Y_{\nu}^{\lambda}(\theta,\phi), \]

we obtain

\[ Q^{(2)}_{\mu} = \sum_{\nu} D_{\mu\nu}^{2}(\theta) Q^{(2)}_{\nu}, \]

where

\[ (EQ6) Q^{(2)}_{\nu} = \sqrt{\frac{16\pi}{5}} \sum_{i=1}^{Z} r_{i}^{(2)}(\theta) \]

\( r_{i}^{(2)} \) are the proton coordinate in the system \( S' \), \( Q^{(2)}_{\nu} \), is the intrinsic electric quadrupole moment.

If

\[ (EQ7) \langle I M K \rangle = \sqrt{\frac{2l+1}{8\pi^2}} D_{MK}^l(\theta) \chi_0 \]

is normalized wave function,

we obtain:

\[ (EQ8) \langle I M^1 K^1 | Q^{(2)}_{\nu} | I M K \rangle = \sum_{\nu} \frac{2l+1}{8\pi^2} \langle \chi_0 | Q^{(2)}_{\nu} | \chi_0 \rangle \int D_{MK}^l(\theta) \delta_{M^1,\nu} \delta_{K^1,\nu} = \sum_{\nu} \]

\( (2l_{M} M | I M^1) (2l_{K} K | I K^1 2\chi_0 | Q^{(2)}_{\nu} | \chi_0) \delta_{M^1,\nu} \delta_{K^1,\nu} \),

(EQ5) if

\[ (EQ6) M = M^1 = I, K = K^1, \]
\[ \text{EQ7) } \mu=0 \text{ and } \nu=0 \]

\[ \text{EQ8) } \text{as the consequence} \]

\[ \text{EQ9) } \langle II K | Q_0^{(2)} | II K \rangle = \langle 21 K | I I K \rangle \langle x_0 Q_0^{(2)} | x_0 \rangle . \]

If \( Q_0 \) is the intrinsic quadruple moment:

\[ \text{EQ10) } Q_0 = \langle x_0 | Q_0^{(2)} | x_0 \rangle \]

and the C. G. coefficient is:

\[ \text{EQ11) } (21D|II) = \frac{I(2I-1)}{\sqrt{I(I+1)(2I-1)(2I+3)}} \]

the relation between the spectroscopic quadruple moment \( Q \) and the intrinsic quadruple moment \( Q_0 \) is:

\[ \text{EQ12) } Q = \langle II K | Q_0^{(2)} | II K \rangle = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0 . \]

Very often \( Q_0 \) is called the “projection factor” and is the connection between the spectroscopic factor \( Q \) and the intrinsic quadruple moment \( Q_0 \).

For the ground state band \((I=K)\). The \( \text{EQ12) } \) becomes:

\[ \text{EQ13) } Q = \frac{I(2I-1)}{(I+1)(2I+3)} Q_0 . \]

For \( K=0 \).

\[ \text{EQ14) } Q = \frac{I}{2I+3} Q_0 \]

In the framework of rotational model it is very easy to establish the connection between \( Q_0 \) and the deformation \( \beta \).

\[ \text{EQ15) } Q_0 = \int e \bar{r} (\vec{r}^1) (3 \hat{Z}^{12} - r^{12}) \delta \tau^1 = \sqrt{\frac{16 \pi}{5}} \int e \bar{r} (\vec{r}^1) r^{12} J_0^{(2)} J_0^{(2)} (\phi^1 y^1) \delta \tau^1 , \]
Where \( \rho_c(\vec{r}) \) is the electric charge density and \( \delta r^1 = r^{12} \delta r^1 \delta \Omega^1 \)

If the charge is uniform, we have:

\[
(\text{EQ16}) \quad \rho_e = \frac{3 e Z}{4 \pi R_0^3},
\]

and the angular dependence of \( R \) is:

\[
(\text{EQ17}) R(\theta^1) = R_0(1 + \beta Y_0^2(\theta^1)).
\]

Finally

\[
Q_0 = \rho_e \sqrt{\frac{16 \pi}{5}} \int_{\{4\pi\}} Y_0^2(\theta^1 \to \varphi) \int_0^{R(\theta^1)} r^{14} \delta r^1.
\]

By integration we obtain:

\[
(\text{EQ18}) Q_0 = \frac{3}{\sqrt{5} \pi} Z R_0^2 \beta (1 + 016 \beta + \ldots).
\]

The ( EQ18 ) allows one to determine the deformation parameter \( \beta \) by the knowledge of \( Q_0 \).

8. Nuclear Deformation

The basic hypothesis is that the nuclear matter is uniformly distributed in a volume \( V \) limited by the surface \( \mathcal{R}=\mathcal{R}(\theta^1, \Phi) \).

As a function of spherical harmonics may be written:

\[
R(\theta, \Phi) = R_0 \left[ 1 + \sum_{\lambda, \mu} \alpha_{\lambda, \mu} Y_{\lambda, \mu}(\theta, \Phi) \right].
\]

For spherical nuclei \( \lambda=2 \) in the system \( S^1 \) only \( \theta^1 \) is able to characterize the nuclear shape.

By the use of Legendre polynomials, we have:

\[
(\text{ND1}) \quad R(\theta^1) = R_0 \left[ 1 + \beta Y_0^2(\theta^1) \right],
\]

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Where $\beta = a_{20}$ and $R_0$ is the mean radius of the nucleons. The deformation parameter $\beta$ is also connected with the ratio

$$\text{(ND2)} \quad \frac{\Delta R}{R} = \frac{a - b}{R},$$

where $a$ is the major axe and $b$ is the minor axe of the ellipsoid

$$\text{(ND3)} \quad Y_0^{(2)}(\theta^1) = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta^1 - 1).$$

we have:

$$\text{(ND4)} a = R_0 (1 + \beta \sqrt{\frac{5}{4\pi}}) \theta^1 = 0$$

$$\text{(ND5)} b = R_0 (1 - \beta \sqrt{\frac{5}{16\pi}}) \theta^1 = \frac{\pi}{2}.$$

As the consequence of (ND4) and (ND5), we have:

$$\text{(ND6)} \quad \beta = \frac{4}{3} \sqrt{\frac{\pi}{5}} \frac{\Delta R}{R_0} = 1.06 \frac{\Delta R}{R_0}.$$

If $\beta > 0$ we have a prolate spheroid, if $\beta < 0$ we have an oblate spheroid.

9. Vibrational model

The vibrational model, similar to rotational model, is a collective description of the even-even nuclei (1, 2, 3, …). The main difference between the two models is that in the rotational model the deformation parameter $\beta$ is a static quantity, in the vibrational model the deformation parameter $\alpha_{\lambda,\mu}$ is a dynamic quantity.

To start with, we consider, for even-even nuclei, only harmonic vibrations, around the spherical shape.

The nuclear surface, as a function of spheric harmonics, is:
(VM1) \[ R(\vartheta, \varphi) = R_0 \left[ 1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_\lambda^\mu Y_\lambda^\mu(\vartheta, \varphi) \right] = R_0 + \Delta R \]

where:

\( R_0 \) is the radius of the spheric nucleus,

\( \alpha_\lambda^\mu \) are the coefficients describing the deformation, in general time depend.

Because the radius \( R \) is a real quantity.

(VM2) \[ \alpha_\lambda^\mu = (-1)^\mu \alpha_\lambda^{-\mu} . \]

In the case of harmonic vibrations, the Hamiltonian operator is:

(VM3) \[ H_\lambda = \frac{\beta_\lambda}{2} \sum_{\mu=-\lambda}^{\lambda} |\dot{\alpha}_\lambda^\mu|^2 \sum_{\mu=-\lambda}^{\lambda} |\alpha_\lambda^\mu|^2 , \]

where \( \beta_\lambda \) and \( C_\lambda \) are connected to the frequency \( \omega \) of the vibration by the relation:

(VM4) \[ \omega = \sqrt{\frac{C_\lambda}{\beta_\lambda}} . \]

In the Hamiltonian (VM3)( harmonic oscillations) there are only quadratic terms

\( |\alpha_\lambda^\mu|^2 \). In order to take into account also unharmonic terms, different contributions can be written:

(VM5) \[ \{ \alpha_\lambda^1 \alpha_\lambda^1 \alpha_\lambda^3 \}, \{ \dot{\alpha}_\lambda^1 \dot{\alpha}_\lambda^1 \alpha_\lambda^2 \}, \ldots \]

9.1. Appendix vibration model

For the sake of completeness we briefly discuss two different modes: the so called “breathing mode” and the Goldhaber and Teller [4] mode.

The “breathing mode” is a spherical density oscillation, with the quantum number
$O^+, T=0$. Its transition density is not only concentrated on the nuclear surface, but involves changes of the nuclear density over the whole volume. Its excitation energy lies at:

\[(VM6) \quad E_0 \approx 8_0 A^{-1/3} \text{MeV}.\]

This energy is a direct measure of nuclear incompressibility. In $^{208}\text{Pb}$ the experimental breathing mode lies at 13.8 MeV, in good agreement with the theoretical predictions.

In the model proposed by Goldhaber and Teller (VM4) the proton sphere vibrates the neutron sphere.

![Figure 6: The model of goldhaber and Teller](image)

The parameters $\alpha_{\lambda i}$ are coupled at zero angular momentum and the numbers of time derivative $\dot{\alpha}_{\lambda i}$ is even, in order to take into account the rotational and the time-reversal invariance.

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By a canonical transformation, we introduce the operators \( b^\mu_\lambda \):

\[
\hat{\alpha}^\mu_\lambda = \sqrt{\frac{\omega_\lambda}{2C}} \left[ b^\mu_\lambda + (-1)^\mu b^\mu_\lambda \right],
\]

(VM7)

\[
\hat{\alpha}^\mu_\lambda = \sqrt{\frac{\hbar \omega_\lambda}{2\beta_\lambda}} \left[ b^\mu_\lambda - (-1)^\mu b^\mu_\lambda \right].
\]

(VM8)

The operators \( b^\mu_\lambda \) have the following commutation rules:

\[
[ b^\mu_\lambda, b^{\mu'}_\lambda ] = 0.
\]

(VM9)

\[
[ b^\mu_\lambda, b^{\mu'}_\lambda ] = \delta_{\mu,\mu'} \delta_{\lambda,\lambda'}.
\]

(VM10)

As function of \( b^\mu_\lambda \) operators, the Hamiltonian \( H_\lambda \) can be written:

\[
H_\lambda = \sum_{\mu=-\lambda}^{\lambda} \frac{\hbar \omega_\lambda}{2} (b^\mu_\lambda b^{\mu'}_\lambda + b^{\mu'}_\lambda b^\mu_\lambda) = \sum_{\mu=-\lambda}^{\lambda} \hbar \omega_\lambda \left( b^\mu_\lambda b^{\mu'}_\lambda + \frac{1}{2} \right).
\]

(VM11)

if

\[
N^\mu_\lambda = b^\mu_\lambda b^{\mu'}_\lambda, N_\lambda = \sum_{\mu=-\lambda}^{\lambda} N^\mu_\lambda,
\]

we have

\[
H_\lambda = \sum_{\mu=-\lambda}^{\lambda} \hbar \omega_\lambda (N^\mu_\lambda + \frac{1}{2}) = \hbar \omega_\lambda (N_\lambda + \frac{2\lambda + 1}{2}).
\]

(VM12)

As a consequence of (VM13), we obtain the energy spectrum of even-even vibrational nucleus.
In particular, the electric quadrupole moment of $2^+$ state $(Q_2)$ is zero.

The wave function of ground state is denoted by $|0\rangle$, with the condition:

\[(VM14) \quad b^0|0\rangle = 0, \langle 0|0\rangle = 1.\]

In order to obtain the wave function of excited states, we apply the creation operator on the ground state:

\[(VM15) \quad |0\rangle, b^\mu|0\rangle, b^\mu^+|0\rangle, b^\mu^+ b^\nu^+|0\rangle, b^\mu^+ b^\nu^+ b^\rho^+|0\rangle.\]

The harmonic description of even-even nuclei isn't in good agreement with the experimental data.

In order to obtain a good agreement with the experimental data, we introduce in the Hamiltonian “unharmonic terms” such as:

\[(VM16) \quad \omega^{30} b^{\mu^+} + \ldots ,\]

where $\omega^{30}, \omega^{31}, \omega^{40}$ are constant, fitted from the experimental data.
As a simple example of unharmonic ity, we report the work of A.K.K. Kerman and C. M. Shakin [Ref 7] in the case of the spectrum $^{62}Ni$.

$$E_{1}/E_{2}^+$$

| $E$(MeV) | $|n|$ |
|---------|-----|
| 2.336   | 4+  |
| 2.302   | 2+  |
| 2.048   | 0+  |
| 1.172   | 2+  |
| 0       | 0+  |

Figure 8: The spectrum of $^{62}Ni$

A cause of the unharmonic term $\left(b_{1}^{x}b_{2}^{x}b_{2}^{y}\right)$, the number of phonons is mixed.

The structure of the wave function is:

$$|0_1\rangle = |00\rangle$$
$$|0_1 = a|12\rangle + \sqrt{1-a^2}|22\rangle$$
$$|2_2 = a|22\rangle - \sqrt{1-a^2}|12\rangle$$
$$|0_2\rangle |20\rangle$$
$$|4_1\rangle = |24\rangle$$

10. Electric quadrupole transitions

If the nucleus is similar to a fluid of uniform charge, we have:

(VM18) $m_{e}(x,\mu) = \int e(r)r^3Y_{\lambda}^{\mu}(\theta,\phi)dr = \int Y_{\lambda}^{\mu}|\theta,\phi|dr \int_{0}^{2\pi} r^{\lambda+2}dr$

(VM19) $e\int |\theta,\phi|dr \int_{0}^{2\pi} r^{\lambda+2}dr = \int r^{\lambda+2}dr$
(VM20) \[ \frac{Q_e}{\lambda+3} \int d r \int Y^\mu_\lambda(\varphi, \phi R^{\lambda+3})|\varphi, \phi \rangle. \]

If

(VM21) \[ R = R_0 (1 + \sum_{\lambda, \mu} \alpha_\lambda \mu Y_\lambda \mu (\varphi, \phi) + \cdots)Q_e = \frac{3Ze}{4\pi R_0^3}, \]

at first order we have:

(VM22) \[ m_e(\lambda, \mu) = \frac{3Ze}{4\pi R_0^3} \int d r Y^\mu_\lambda(\varphi, \phi) (1 + R^{\lambda+3}) \sum_{\lambda, \mu} \alpha_\lambda \mu Y_{\lambda, \mu} (\varphi, \phi) + \cdots, \]

(VM23) \[ m_e(\lambda, \mu) = \frac{3Ze}{4\pi} R_0^3 \alpha_\lambda \mu. \]

For quadrupole transition we have:

(VM24) \[ m_e(2, \mu) = \frac{3Ze}{4\pi R_0^2} \sqrt{\frac{\hbar}{2B\omega}} \left(b_\mu^\dagger + (-)^\mu b_\mu \right). \]

From the structure of \( m_e \) operator it is very easy to calculate its matrix elements.

All the diagonals matrix elements are zero and the selection rules are:

(VM25) \[ |\Delta l| \leq 2, \quad |\Delta n| \leq 1. \]

11. Particle-vibration coupling

In 1967 B.R. Mottelson proposed a simple mechanism to handstand the unharmonicity in the atomic nuclei the particle vibration coupling.

The Hamiltonian of the model is:

(VM26) \[ H_I = H_S + H_P + H_{int} , \]

where

\( H_S \) describes the vacillations of even-even nucleus,
\( H_p \) is the Hamiltonian of single-particle

\[
(\text{VM27}) \quad H_p = T + V(r),
\]

where \( T \) is the kinetic energy of the single-particle and \( H_{int} \) is the coupling between the surface oscillations of the even-even nucleus and single particle.

If we take into account only linear term in \( H_{coup} \) and quadrupole \( (\lambda = 2) \) and octupole \( (\lambda = 3) \) oscillations for the even-even nucleus, we have:

\[
(\text{VM28}) \quad H_{coup} = K(r) \sum_{\lambda = 2}^{3} (\alpha_{\lambda}^e Y_{\lambda}),
\]

where

\[
K(r) = \text{Vor} \frac{df}{dr}, \quad f(r) = \frac{1}{1 + e \frac{r - R}{\mu}}.
\]

Eigenvalues and eigenvectors of the coupled system are obtained by the standard diagonalization procedure on the basis \( \{|n \frac{1}{2} e\}, \{N2 R2, N3 R3\} \), where \( j \) and \( R \) are the angular momentum of the single-particle and of the states \( (N2, N3) \) of the even-even nucleus. \( I \) is the total angular momentum of the add nucleus.

The matrix elements of \( H_{coup} \) are:

\[
(\text{VM29}) \quad \langle j(N2 R2, N3 R3) R; \mathcal{H}\{H_{coup}\} | j' (N' 2 R' 2, N' 3 R' 3) R'; \mathcal{H}\rangle = \sum_{\lambda = 2}^{3} \langle n I j | k(r) | n' I' j' \rangle \gamma(-1)^{R_j + R_j' + \lambda + J + 1/2}
\]

where

\[
\gamma[2 if \lambda = 3] \quad \gamma[3 if \lambda = 2]
\]
In this model there are different parameters:

1) $\varepsilon_j$ : single particle energies

2) $\hbar \omega_{\lambda}$ : energies of the oscillations of even-even nucleus

In general also the matrix elements $\langle nlj|k(r)|n'l'j'\rangle$ are fitted on the experimental data.

In order to avoid to use free parameters, the following procedure has been adopted:

1) the single particle $\varepsilon_j$ are obtained by direct reactions,

2) the energies $\hbar \omega_{\lambda}$ are extracted by the even-even nucleus,

3) the radial wave function $R_{nlj}$ is obtained by the potential

\[ V(r) = -V_0 f(r) + V_{SO} \lambda_n^2 \frac{1}{r} \frac{d}{dr} \sigma \cdot e + V_{coup} \]  

The parameters of the potential $V(r)$ are taken from K. Bleuler et al [Ref 8].

As example we apply the model to the nuclei:

\[ z_{58}^{\text{Ni}} - p \rightarrow ^{57}\text{Co} , z_{50}^{\text{Sn}} - p \rightarrow ^{115}\text{In} , z_{82}^{\text{Pb}} - p \rightarrow ^{209}\text{Tl} \]

This choice is followed for different reasons:

1) the even-even nuclei have protons closed shells (28-50-82) and a few neutrons in the single particles orbits. The simplest microscopic description of states $2^+, 4^+ \ldots$ is based on quasi-neutron description [Ref 9]. In the coupling of these states with a proton, the antisymmetrization procedure is not very important. We assume also that the role of the partial level occupation is not basic at all.

To start with, we consider the case of $^{57}\text{Co}$ and $^{115}\text{In}$.

1) $z_{58}^{\text{Ni}} - p \rightarrow ^{57}\text{Co}$
The collective low-lying states of $^{57}\text{Co}$ are: $J^m=3/2^+,5/2^+,7/2^+,9/2^+,11/2^+$. The splitting of the multiplet (-0.7 MeV) is not small at all and then the particle-vibration coupling is not weak.

The gravity centre of the multiplet is 1.65 MeV and $E(2^+)^{58}\text{Co}=1.45$ MeV. This imply a mixing of different phonons number. In ref [ ref 10 ] the model space is $1f_{7/2},2s_{1/2}$, and $2d_{3/2}$ (for proton hole) and tree quadrupole fonons and one octupole.

The single particle energies are taken from A.G. Blair and A.D. Armstrong [ Ref 11 ] and $\hbar\omega_2=1.45 \text{MeV}$, $\hbar\omega_3=4.40 \text{MeV}$ See table

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Core</th>
<th>$\hbar\omega_2$</th>
<th>$\beta_2$</th>
<th>$\hbar\omega_3$</th>
<th>$\beta_3$</th>
<th>$\beta_1/2$</th>
<th>$\beta_3/2$</th>
<th>$\beta_1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{57}\text{Co}$</td>
<td>$^{58}\text{Ni}$</td>
<td>1.45</td>
<td>0.19</td>
<td>4.4</td>
<td>0.13</td>
<td>0</td>
<td>3.31</td>
<td>2.36</td>
</tr>
<tr>
<td>$^{59}\text{Co}$</td>
<td>$^{60}\text{Ni}$</td>
<td>1.33</td>
<td>0.21</td>
<td>4</td>
<td>0.11</td>
<td>0</td>
<td>2.98</td>
<td>2.12</td>
</tr>
</tbody>
</table>

$\eta_\lambda=\frac{\beta_\lambda}{\sqrt{(2\lambda+1)\pi}}$


We compare the experimental spectrum of $^{57}\text{Co}$ with the theoretical one.

| Nucleus | $\langle if_{7/2}|K(r)|if_{7/2}\rangle$ | $\langle if_{7/2}|K(r)|ld_{3/2}\rangle$ | $\langle if_{7/2}|K(r)|2s_{1/2}\rangle$ |
|---------|--------------------------------|--------------------------------|--------------------------------|
| $^{57}\text{Co}$ | 58.40 | 48.27 | -49.96 |
| $^{59}\text{Co}$ | 58.80 | 48.23 | -50.25 |

| Nucleus | $\langle ld_{3/2}|K(r)|ld_{3/2}\rangle$ | $\langle ld_{3/2}|K(r)|2s_{1/2}\rangle$ | $\langle 2s_{1/2}|K(r)|2s_{1/2}\rangle$ |
|---------|--------------------------------|--------------------------------|--------------------------------|
| $^{57}\text{Co}$ | 41.63 | -40.09 | 44.24 |
| $^{59}\text{Co}$ | 41.30 | -40.05 | 44.38 |
In table we compare the theoretical transitions probability with the experimental ones.

Single- hole strength

<table>
<thead>
<tr>
<th>$^{57}\text{Co}$</th>
<th>$^{59}\text{Co}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref$^{(1)}$</td>
<td>Present calc.</td>
</tr>
<tr>
<td>$r_{7/2^-}$</td>
<td>$0.69$</td>
</tr>
<tr>
<td></td>
<td>$0.52$</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$0.83$</td>
</tr>
<tr>
<td></td>
<td>$0.43$</td>
</tr>
</tbody>
</table>


The experimental values are obtained

K.S. Burton and L.C. Mc Intyre, from the reaction $^{54}\text{Fe}(\alpha,\rho,\gamma)^{57}\text{Co}$. As can be seen, for the transition $B(E2; 11|2\rightarrow0\rangle$ the agreement between the experimental and the theoretical one is not good at all.

2) We discuss here the case $^{116}\text{Sn} - \rho - ^{115}\text{In}$. The comparison between the theoretical calculation and the experimental data is quite good.

As it is well known, the collective properties of $2^+$ states of even-even Sn isotopes are a smooth function of A.

As a consequence, also the collective states of odd In isotopes are quasi similar ($^{113}\text{In}$ and $^{117}\text{In}$).

The model space for $^{115}\text{In}$ is to one hole state $1g_{9/2}$, $2\rho_{1/2}$, $2\rho_{3/2}$ and $1f_{5/2}$ and

$\lambda = 2$ and $\lambda = 3$ for the even-even nuclei. [Ref 10].
Because the experimental data are not able to determine in an unique way the single particle levels, we have adopted a fitting procedure of spacing \( \Delta_j \equiv \varepsilon_j - \varepsilon_{1g_{9/2}} \)

\[ (j = 2_{p2/3}, 2_{p3/2}, 1_{f5/2}) \]

The comparison of the experimental spectrum of \( ^{115}In \) with the theoretical one is quite good.

Four states, with \( I^n = 5/2^+, 7/2^+, 9/2^+, 11/2^+, 13/2^+ \), strangely excited an elastic scattering of deutons and coulomb excitation, are interpreted as the member of a quinted, obtained by the coupling of the proton hole \( 1g_{9/2} \) with the quadrupole oscillations of \( ^{116}Sn \).
In fig. 9 it is very easy to note that the member of the multiplet are well reproduced by the model.

**Conclusions**

As discussed in the previous sections, even if the particle vibration coupling is a very simple model, it is also able to explain the energy spectrum, the transition probabilities, the spectroscopic functions and so on.

But there are some limitations and difficulties in the use of this model.
1) the first one is the determination of single particle energy $\varepsilon_j$ and the deformation parameters $\beta_\lambda$.

2) the second one is the coupling of the even-even nucleus $|\xi\downarrow\rangle$ with a hole A-2 nucleus $\uparrow\uparrow$.

If we take into account both the contributions we have:

$$\alpha|\xi\downarrow\sim\beta\downarrow\uparrow$$

Very often the second contribution is very important.

![Figure 10: Coupling between different excitations](image_url)

**References**


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