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A Class of Automata Networks for Diffusion of Innovations Driven by Riccati Equations

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Renato Guseo
Department of Statistical Sciences
University of Padua
Italy

Mariangela Guidolin
Department of Economics
University of Padua
Italy

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Keywords: diffusion process, Bass model, communication network, cellular automata, Riccati equation.

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Department of Statistical Sciences
Via Cesare Battisti, 241
35121 Padova
Italy

tel: +39 049 8274168
fax: +39 049 8274170
http://www.stat.unipd.it

Corresponding author:
Renato Guseo
tel: +39 049 827 4146
renato.guseo@unipd.it
http://www.stat.unipd.it/guseo/

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1 Introduction

Since the publication of the Bass model in 1969, research on diffusion of innovation and innovation theory have raised a growing interest, with reference both to consumers behaviour (see Gatignon and Robertson (1985)) and marketing management for developing new strategies focused on potential adopters. Interesting reviews of the literature on diffusion models are provided by Mahajan and Muller (1979), Mahajan *et al.* (1990), Mahajan *et al.* (2000) and Meade and Islam (2006) where it is highlighted that the purpose of the diffusion model is to describe the successive increases in the number of adoptions and predict the continued development of a diffusion process already in progress. In spite of the more recent research proliferation in this field, the basic known diffusion models are those of Fourt and Woodlock (1960), Mansfield (1961) and Bass (1969). The last one results from the summation of the other two and assumes that potential adopters are influenced in their purchase behaviour by two sources of information: an external, like mass–media communication and an internal, word-of-mouth. Furthermore, it is assumed that adopters can be influenced only by one of these two forces, forming two distinct groups, innovators (mass-media) and imitators (word-of-mouth) and therefore, part of the adoption is based on learning by imitation and part of it does not. Formally, the model can be expressed through a first order differential equation

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t)). \tag{1}$$

Instantaneous adoptions, $z'(t)$, are proportional to the residual market $(m - z(t))$ and determined by two additive components. The first one, $p(m - z(t))$ refers to innovators, who adopt with a rate p called coefficient of innovation. The group of innovators is surely crucial for the "take-off" of diffusion, even if present at any stage of the process.

The second part of Equation (1), $qz(t)/m(m - z(t))$, represents adoptions of buyers who are influenced by previous adopters (word-of-mouth effect, w–o–m for short) through parameter q. The effect of parameter q is modulated by the ratio $\frac{z(t)}{m}$, which at time $t = 0$ is clearly zero, $\frac{z(t)}{m} = 0$, justifying the temporal delay of adoptions due to w–o–m effect. As a consequence, if innovators are necessary for the initial phase of the diffusion process, imitators are crucial for its development and growth, the life cycle of an innovation depending on these two combined effects.

An extremely useful extension of the Bass model is represented by the Generalized Bass Model (GBM) by Bass et al. (1994) allowing to include the presence of exogenous interventions (strategic interventions, policies, marketing strategies). The GBM equation is

$$z'(t) = \left(p + q\frac{z(t)}{m}\right)(m - z(t))x(t), \tag{2}$$

where $x(t)$ denotes a quite general intervention function, whose effect can accelerate or delay adoptions over time but cannot control independently the potential market m or the intrinsic diffusion parameters p and q.

Indeed, one of the main assumptions in the Bass models relates to the potential market (or carrying capacity) m whose size is considered fixed along the whole diffusion process. One can see this aspect by inspecting both Equations (1) and (2). In this paper we propose a modification of this assumption developing a model in which the potential market is no longer constant but a function of time, $m(t)$. A central question requires to motivate this time dependence, presenting a theoretical explanation of a dynamic potential. An evolutionary perspective may offer an appropriate framework.

According to the Bass model, the diffusion of an innovation in a social context is represented as a learning process, in which few persons decide to adopt on the basis of an external information and the others get the relevant information from previous adopters, imitating their behaviour. However, the data we use for modelling a diffusion process do not provide this distinction explicitly, just telling us how much has been purchased at a certain time. Thus, the existence of these groups does not emerge from a direct inspection of data, but, as a working hypothesis on latent categories, it has proven to be an excellent modelling choice in most cases.

Moreover, it suggests some considerations on the role of information heterogeneity for explaining different attitudes in consumption. As we have seen, the Bass model proposes a simple and efficient bipartition of consumers' behaviour based on information channels. Of course we are not saying anything new if we point out the relevance of information for any economic action.

However, starting from a basic level of reasoning, according to which a consumer adopts after being informed about an innovation (its existence and its features), we could investigate more in detail the relationship between information and innovation diffusion.

This is certainly a crucial issue for understanding both individual and collective action within innovation contexts. A relevant contribution, more in qualitative terms, on this topic has been given by Cohen and Levinthal (1990), that defined the concept of *absorptive capacity.* Even though the authors' focus is the firm, we think that very similar considerations may be easily applied in a consumption perspective.

Considered both at the individual and organizational level, the term absorptive capacity refers to the "ability to recognize the value of new information, assimilate it and apply it", Cohen and Levinthal (1990).

It is argued by the authors that this ability to assimilate and exploit a novelty is function of a *prior related knowledge.* That is, the presence of a background of relevant knowledge implies a greater receptiveness to new ideas.

Cohen and Levinthal use this concept both for individuals and organizations. As they point out, in the individual case, this ability is related to cognitive functions of the single person, while to understand an organization's absorptive capacity it is necessary to focus on its communication structure, since this capacity for organizations is not the simple sum of those of its components, but has to do with knowledge transfers.

The concept of absorptive capacity in organizations is particularly interesting for the purposes of this paper, in which we focus on innovation phenomena at the aggregate level.

The adoption of an innovation in a specific social context may be viewed as a direct evidence of an existing absorptive capacity: in fact, the ability to assimilate and accept a novelty may find a simple check in the observed adoption process. Specifically, the potential market m may represent a measure of this absorptive capacity. As we know, the potential market m is typically considered a constant quantity over time.

However, the concept of absorptive capacity suggests a different perspective for considering this aspect.

Since the ability to assimilate an innovation depends on the *accumulation of a prior knowledge*, we could try to define the potential market accordingly. A process of accumulation of knowledge in a social system requires the transfer of information among the components of the system. In this sense Cohen and Levinthal highlight the importance of designing the communication structure of an organization to understand its absorptive capacity. Accumulating knowledge involves some learning dynamics, whose description, in our view, is best reached through an evolutionary model, rather than a cross–sectional modelling, as proposed by Cohen and Levinthal (1990).

Developing an evolutionary perspective, we find convenient to represent a communication structure as a set of informational linkages among the units of the system. As individual knowledge is created connecting ideas and concepts between them but also destroying some existing connections, the development of a *collective knowledge* can be thought of as an *evolving network*, in which some linkages exist, some rise and some others die.

Considering the potential market $m(t)$ as a function of this knowledge process, will imply to make it dependent on a network of connections that changes over time.

Recent studies (see for instance, Mahajan *et al.* (1984); Eliashberg *et al.* (2000)) have confirmed that internal communication forces (w-o-m, learning) play a key role in new product adoption. Goldenberg *et al.* (2001) have noticed that the growing use of the Internet, allowing a very quick and simple spread of information, has raised a new kind of w–o–m called "internet word–of–mouth". In fact, companies are currently investing much effort in viral marketing (Oberndorf (2000)) and today's managers are attending to the power of w-o-m, trying to "manage rather than direct it" (Goldenberg *et al.* (2001). However, little is known on how this interpersonal communication is structured and realized. Actually, the Bass model, that has proven to be very flexible and reliable in forecasting, does not provide a clear explanation on the process of communication underlying adoptions. This probably relates to the aggregate nature of the model. But innovation theory states that "diffusion theory's main focus is on communication channels" (Mahajan *et al.* (1990)) and for this reason their actual structure should be analyzed and understood as much as possible.

Goldenberg *et al.* (2001) say that the gap of knowledge may be linked to the complexity of the w-o-m process, which may be described as a "complex adaptive system", i.e., a system consisting of many interacting agents, whose relations at the micro-level generate emergent, collective behaviour, visible at the macro-level of inquiry. If the Bass model is generally able to capture this macro-behaviour through three parameters (m, p, q), the analysis of the underlying micro-interactions is left to other kinds of models (see, for instance, Chatterjee and Eliashberg (1990) and Roberts and Lattin (2000)) within diffusion of innovation theories in quantitative marketing and methods dealing with the issue of *complexity*. Many scientific disciplines, such as physics, biology and ecology have developed models to investigate how complex systems evolve. Within these, Stochastic Cellular Automata models seem to be a useful choice for connecting behaviour at the micro and macro levels. The perceived complexity of organizations and markets, in which many agents interact with each other, has suggested the use of Cellular Automata also in economic and social fields (see, for instance, Goldenberg and Efroni (2001), Moldovan and

Goldenberg (2004), Goldenberg *et al.* (2005)).

A Cellular Automaton consists of a finite number of individuals (or cells) that interact in a defined environment. Each cell can assume a particular state (for example, adopter, neutral) depending on its state in the previous period of time and on the information received interacting with other cells. The evolution of the state of each cell is controlled by a predefined function called transition rule, which explicitly considers these interactions. The advantage provided by Cellular Automata models is the opportunity to observe the evolution of a given structure through the analysis of every single interaction between its components, representing another way, with respect to aggregate models, to deal with structural change and evolution. In this sense, Cellular Automata models may be powerful complements, rather than complete substitutes, of aggregate models for the analysis of life cycles and evolutionary patterns. In particular, the micro-level descriptive power of Cellular Automata could represent the conceptual introduction for new possible generalizations of the Bass model (see, for instance, Guseo and Guidolin (2006)).

In this paper we use a Cellular Automata Network for describing a network of interacting agents, who communicate between them information about a particular innovation. Thus, we propose a two–phase modelling, representing first the Communication Network and then the proper adoption process that can occur only when there is sufficient knowledge about the involved innovation. In this case, the analysis unit for Cellular Automata is represented by each communication channel (edge) between two agents, whose state can be already active, susceptible of activation, inactive. We suppose that the activation of an edge can occur through a standard w–o–m or imitative process. Moreover, we assume that in the case of very closely related cells, the edge may be activated by an external source of information, such as advertising. In such a way, we are able to describe two distinct behavioural patterns, the imitative and the innovative, reproducing the Bass framework. Furthermore, we consider the possibility of edges' inactivation. This may happen with a natural and autonomous decay process or through a negative word-of-mouth due to resistance to innovation effects. This represents a typical reaction to innovation for dissatisfaction or inadequate performance, whose effect may affect *dynamically* the potential market (see, for instance, Moldovan and Goldenberg (2004)).

All these possibilities are described in a unique transition rule, able to represent the changing state of each edge. Once defined this Communication Network, the second stage of the model relates to the structure of the embedded adoption process.

The paper is organized as follows. In Section 2 we present a stochastic evolution of a Communication Network, extending the binary Automata Network proposed by Boccara *et al.* (1997). In particular, Boccara *et al.* (1997), Boccara and Fukś (1999) and Boccara (2004) proposed, among others, interesting representations of special Automata models, allowing a "Mean Field Approximation". In Section 3 we present a model for a co-evolutive adoption process using a "Mean Field Approximation" to link our Automata Network and an adoption process within a Riccati equation.

The closed form solution of a special non autonomous Riccati equation, which, under particular constraints, provides the standard Bass model and the Generalized Bass Model (GBM) submitted to an environmental intervention function $x(t)$, is proposed in Appendix A . In Section 4 we apply previous results to the co–evolutive model and examine statistical aspects concerned with inference and applications. Section 5 is devoted to an application within bank services. Final comments and discussion are considered in Section 6.

2 Evolution of Knowledge in a Communication Network

Let $G = (V, E)$ be a *finite* directed graph, where $V = \{1, 2, \cdots, i, \cdots, N\}$ is a set of vertices whose cardinality is $N = \mathbf{c}(V)$. The set E of ordered pairs (i, j) called *directed edges* or arcs, $E \subset V \times V$, depicts a subset of all the possible binary relationships within vertices V including reflexive relationships. Due to possible limitations on *connectivity*, the cardinality of E is $U = \mathbf{c}(E) \leq N^2$. From now on, we will use the simpler term *edge* to refer to a proper *directed edge*.

In social or physical systems these constraints may have natural interpretations based on large distances or accessibility censoring limits that *a priori* exclude a possible link between two vertices. Each edge in E may assume, at time t, a special state among a finite set of levels, $Q = \{0, 1, 2, \cdots, K\}$. We will assume a simple binary version, $Q = \{0, 1\}$, i.e., an edge may be active, 1, when an information about an innovation is *transmitted* between vertices of an admissible edge or not, 0. We denote the *state* of an edge (i, j) at time t with an indicator function $c(i, j; t)$. Function $c(i, j; t)$ equals 1 if and only if the edge (i, j) is active, otherwise is zero, in particular, if $(i, j) \notin E$.

The active state of an edge may be *reversible*. With susceptible *reflexive edges* or with strongly connected vertices we represent the possible support of initializing dissemination of information due to a high level of individual specific *prior related knowledge* (using Cohen and Levinthal's terminology) and to external channels of communication like mass media.

Here we follow, only partially, some notations expressed for Automata Networks in Boccara *et al.* (1997) and in Boccara and Fukś (1999).

Let us define a rectangular centered neighborhood $A_{(i,j)}$ around an edge (i, j) with *radii* ${}_1e_i$ and ${}_2e_j \in I\!N$ (the set of natural numbers including 0), i.e.,

$$A_{(i,j)} = \{(r, s) | i - {}_1e_i \leq r \leq i + {}_1e_i, j - {}_2e_j \leq s \leq j + {}_2e_j\}.$$

We assume that the transition rule $g(\cdot)$ governing network states is a function, possibly with stochastic components, of the arc states of the neighborhood $A_{(i,j)}$ of an edge $(i, j) \in E$, i.e., in expanded form,

$$\begin{aligned} c(i, j; t + 1) \quad &= \quad g(c(i - {}_1e_i, j - {}_2e_j; t), c(i - {}_1e_i + 1, j - {}_2e_j; t), \cdots \\ &\cdots \quad , c(i + {}_1e_i, j + {}_2e_j - 1; t), c(i + {}_1e_i, j + {}_2e_j; t)), \end{aligned} \tag{3}$$

where $c(r, s; t) = 0$ if $(r, s) \notin E$. We assume here a discrete time $t \in I\!N$.

We may specify function $g(\cdot)$ by a combination of local and individual effects. A prominent local effect on the state $c(i,j;t+1)$ of an edge (i,j) is determined by the joint influence of neighboring edge states. More precisely, we define a kind of *local pressure* (probability) of the system, $\sigma_c(i,j;t)$, upon edge (i,j) to turn from an uninformative status towards an informative one. This pressure depends on a flexible probability measure, $p_{n,m} \geq 0$, that allows a more general description of a neighborhood, possibly (i,j)–dependent.

$$\sigma_c(i,j;t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c(i+n, j+m; t) p_{n,m}\, ; \qquad \sum_{n,m} p_{n,m} = 1. \tag{4}$$

If we assume that this local pressure is *translational invariant*, we may consider the "Mean Field Approximation" that excludes the local effect of distribution $p_{n,m}$,

$$\sigma_c(i,j;t) \simeq \nu(t) = \sum_{i,j} \frac{c(i,j;t)}{U}. \tag{5}$$

Note that if a censoring constraint uniformly acts outside a neighborhood of given pattern, relationship (5) must be weakened with a correction, $v < 1$,

$$\sigma_c(i,j;t) \simeq v\nu(t), \tag{6}$$

where v represents a *spatial memory depth* or, in other terms, only the "visible" fraction – assumed (i,j)–independent – of the distribution $p_{n,m}$.

Let us define now a particular rule $g(\cdot)$, through a partially probabilistic specification, in order to describe some interesting components of individual and local information diffusion,

$$\begin{aligned} c(i,j;t+1) &= c(i,j;t) + Bi(1,p_c)\, I_{(c(i,j;t)=0)} + \\ &+ Bi(1, q_c\, \sigma_c(i,j;t))\, I_{(c(i,j;t)=0)} + \\ &- Bi(1,e_c)\, I_{(c(i,j;t)=1)} - Bi(1, w_c\, \sigma_c(i,j;t))\, I_{(c(i,j;t)=1)}. \end{aligned} \tag{7}$$

Notice that this transition rule must be interpreted within the conventional notations of Computer Science in a sequential order from the left to the right. For instance, the indicator function $c(i,j;t)$ may change its status, "within time t", if the second addend turns out to be 1 and similarly for the subsequent components. Only after the last additive term the obtained result is transferred (=) to the left hand member $c(i,j;t+1)$ and indexed by time $t+1$.

The second component of Equation (7), $Bi(1,p_c)I_{(c(i,j;t)=0)}$, depends upon a binomial experiment, with parameter p_c, which is realizable only if the indicator function $I_{(c(i,j;t)=0)}$ is set to one, i.e., proposition $(c(i,j;t)=0)$ is true. The meaning of this first component may be linked to the direct effect of external information like mass media communication channels and the change of state is possible, with probability p_c, only if "institutional communication" reaches *susceptible* edges, i.e., reflexive edges or strongly connected vertices.

The third component of Equation (7) considers the joint probability $q_c\sigma_c(i,j;t)$, that depicts the *local pressure* of neighboring knowledge, $\sigma_c(i,j;t)$, and the intrinsic attitude of pure imitative response pushed by a binomial parameter q_c. This second experiment is an opportunity strictly referred to standard edges (not reflexive or weakly connected) and expresses the common perceived fact that *imitative behaviour* is an *individual attitude* based upon a *local* geometry of evidence.

Note that the activation of these two components is strictly alternative or exclusive, i.e., if the first experiment changes the status of an edge, the second one is switched off and vice versa: the activation of an imitative relationship forbids the innovative behaviour.

The fourth component is a *decay effect* driven by a binomial $Bi(1,e_c)$ under the control of the correct state, $I_{(c(i,j;t)=1)}$, and describes the possible withdrawal from an active state representing a normal loss of information.

The fifth component is a negative word–of–mouth driven by a binomial $Bi(1,w_c)$ under the control of the correct state, $I_{(c(i,j;t)=1)}$, and represents the forced withdrawal from the active state due to the opposite effects of local pressure producing resistance to innovation. Also these two exit rules are strictly alternative.

Here we suggest a useful interpretation of the proposed stochastic transition rule (7) with reference to some contributions in literature on social networks theory. In particular, the distinct roles assigned to strongly connected vertices, on the one hand, and standard edges (weakly connected), on the other, may be fruitfully related to the theory of strong and weak ties formulated by Granovetter (1973). In his work "the Strenght of Weak Ties" (1973) Granovetter highlighted that persons are often influenced by others with whom they have weak relationships, called "weak ties" to distinguish them from those "strong ties" that are stable and frequent linkages, which create individuals' strictly personal networks. We may consider weak ties as the crucial factor for the spread of information by word-of-mouth as it is also highlighted in Rogers (2003). Goldenberg *et al.* (2001) claimed that "the significance of weak ties lies in their potential to unlock and expose interpersonal networks to external influences (individuals in distant networks), thus paving the path for the spread of information throughout society". Thus, we may conclude that diffusion of knowledge in a social system mostly depends on the presence of these weak ties. Strong ties constitute those intimate relationships whose role may be better related to an (eventual) innovative behaviour. In the transition rule (7) we have also considered the possibility for an edge to be inactivated by a natural decay process or by a negative word–of–mouth, exactly with the same logic followed for the positive diffusion of information.

Once defined the stochastic transition rule (7) informing on how an edge may be activated, the second step is to recognize a convenient method to infer that emergent collective behaviour we are interested in.

In general, Cellular Automata are implemented through computer simulations generating a global behaviour from an individual (local) rule. The use of such techniques raises evident questions about the reliability of selected simulation parameters for which information is usually not available (see, for more details, Guseo and Guidolin (2006)).

Directly facing with this problem we alternatively propose a local to global mapping considering a "Mean Field Approximation" of the transition rule (7). In this way we may statistically infer collective behaviour from historical observed aggregate data.

Let us consider, therefore, the average number of active edges within E at time t following the mean behaviour of the transition rule (7),

$$U\nu(t+1) = U[\nu(t) + p_c(1-\nu(t)) + q_c\nu(t)(1-\nu(t)) - e_c\nu(t) - w_c\nu^2(t)]. \tag{8}$$

Note that if we incorporate truncating effects like those described in Equation (6), parameter q_c collects two unidentifiable effects, the spatial memory depth v and the intrinsic pure imitative effect q: $q_c = vq$.

We can approximate previous discrete time equation with a continuous Riccati equation, namely,

$$\nu'(t) = -(q_c + w_c)\nu^2(t) + (q_c - p_c - e_c)\nu(t) + p_c, \tag{9}$$

and if we skip e_c and w_c components, we obtain a standard Bass (1969) model.

Solution $\nu(t)$ of previous Equation (9) is described in Appendix A as a special case for $f(\cdot) = g(\cdot) = 1$ and its explicit form is discussed in Section 4.

Potential market (carrying capacity) definition

We conclude this section highlighting the important modelling choice related to the communication network we have designed. Function $U\nu(t)$ defines an aggregate temporal evolution of the knowledge or the awareness of an innovation within the proposed communication network. Such a knowledge, based on active edges, is only a preliminary step in absorptive capacity definition following Cohen and Levinthal's (1990) terminology. We are interested in transforming this dynamic knowledge in a dynamic carrying capacity or potential market in order to define a potential boundary for the nested adoption process. This potential boundary is not a function of observed quantities: it is a latent structure that we can not measure directly.

The positive squared root of $U\nu(t)$,

$$k(t) = \sqrt{U}\sqrt{\nu(t)}, \tag{10}$$

depicts the upper bound of the *carrying capacity* $m(t)$ for the related process of innovation adoption by individuals describing the system, here represented as vertices of the graph $G = (V, E)$. Note that $k(t)$ is proportional to $\sqrt{\nu(t)}$, so that we can assume

$$m(t) = K\sqrt{\nu(t)} \tag{11}$$

as the actual carrying capacity, where $K \leq \sqrt{U}$ if a vertex does not replicate an adoption. If replication of adoption is allowable, K may be much greater than $\sqrt{U}$.

An extension in $U\nu(t)$ transformation may be based on $\nu(t)^{\alpha}$ in order to take into account possible dimensional collapse of $E \subset V \times V$.

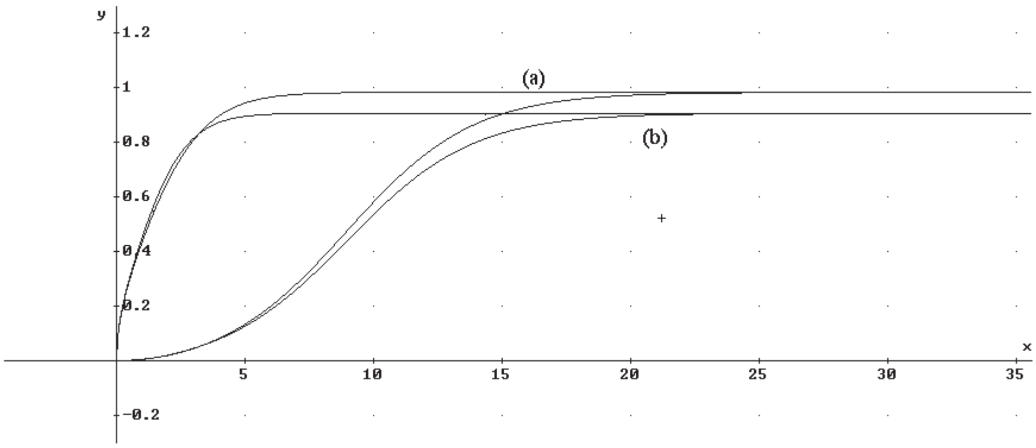

Figure 1: Two different communication frameworks. Common adoption parameters: $q_s = 0.4,\ p_s = 0.01,\ r_s = 0$; Common communication parameters: $K = 1,\ p_c = 0.15,\ e_c = 0.03$. Special cases: Case $(a) : q_c = 0.7,\ w_c = 0$, Case $(b) : q_c = 0.9,\ w_c = 0.2$.

3 Co–evolution of the Diffusion of an Innovation

We denote the state of a vertex $i \in V$ at time t with the indicator function $s(i;t)$. Following the same guidelines developed in Section 2, we define a transition rule for the description of an individual adoption process over time with the notation of cellular automata, i.e.,

$$\begin{aligned} s(i;t+1) &= s(i;t) + Bi(1,p_s)\, I_{(s(i;t)=0)} + \\ &+ Bi(1, q_s\, \sigma_s(i;t))\, I_{(s(i;t)=0)} + \\ &- Bi(1,r_s)\, I_{(s(i;t)=1)} + \\ &+ s(i;t).\frac{m'(t)}{m(t)}. \end{aligned} \tag{12}$$

The first four additive components of the left hand member in Equation (12) may be interpreted following the same ideas of the previous section and the conventional notation is interpreted, as in Equation (7), sequentially, following Computer Science updating rules “within time t”. The result is transferred (=) to the left hand member $s(i;t+1)$ and indexed with time $t+1$.

In particular, the second component, $Bi(1,p_s)\, I_{(s(i;t)=0)}$, represents the direct effect of mass media. Experiment $Bi(1,p_s)$ is performed with adoption innovative probability p_s if $s(i;t) = 0$. The third component represents the w–o–m contribution to adoption under a joint imitative probability based on two factors, an imitation coefficient, q_s, and a specific local pressure stimulating imitative adoption, $\sigma_s(i;t)$. The fourth component represents a decay exit rule with exit probability r_s. The fifth component, $s(i;t)\cdot\frac{m'(t)}{m(t)}$, describes an infinitesimal variational contribution to the individual state due to the relative varying effect of carrying capacity $m(t)$ over time and is independent of K. Note that for a constant *carrying capacity* $m(t)=M$, this component gives a null contribution. This infinitesimal contribution depicts $s(i;t)\cdot\frac{m'(t)}{m(t)}$ as an interaction of the individual state with the global increasing or shrinking effect of the potential market as a function of knowledge network dynamics. A possible extension may be based on a suitable weighting of the above interaction, i.e., $\alpha s(i;t)\cdot\frac{m'(t)}{m(t)}$. In the sequel we assume $\alpha=1$.

The average behaviour of Equation (12) followed by a summation over all the states $s(i;t)$ within V is a discrete time *co–evolutive model*

$$y(t+1)=y(t)+p_s(m(t)-y(t))+q_s\frac{y(t)}{m(t)}(m(t)-y(t))-r_sy(t)+y(t)\frac{m'(t)}{m(t)}. \quad (13)$$

A continuous approximation of previous Equation (13) is

$$y'(t)=m(t)\left\{-r_s\frac{y(t)}{m(t)}+\left(p_s+q_s\frac{y(t)}{m(t)}\right)\left(1-\frac{y(t)}{m(t)}\right)\right\}+y(t)\frac{m'(t)}{m(t)}. \quad (14)$$

Perturbed evolution of an adoption process

An extension of the previous representation is based on the modification of uniform dynamics due to exogenous interventions effects during the diffusion process. A similar approach is developed in Bass et al. (1994) with the Generalized Bass Model (GBM).

We model this more flexible context multiplying by an impact function, $x(t)$, whose neutral level is obviously $x(t)=1\,\forall t$, i.e.,

$$y'(t)=m(t)\left\{-r_s\frac{y(t)}{m(t)}+\left(p_s+q_s\frac{y(t)}{m(t)}\right)\left(1-\frac{y(t)}{m(t)}\right)\right\}x(t)+y(t)\frac{m'(t)}{m(t)}. \quad (15)$$

Remind that $x(t)$ exerts its effect only on the future and, therefore, on the first component of Equation (15) which is a function of the residual market.

This is a special Riccati equation analyzed in Appendix A. Note that in original GBM we have two special constraints: decay component is excluded, $r_s\,y(t)/m(t)=0$, and the potential market (carrying capacity) is constant, $m(t)=M$. The solution of Equation (15) is presented in Section 4 under the pertinent substitutions, in particular, $f(\cdot)=x(\cdot)$ and $g(\cdot)=m(\cdot)$.

4 Statistical Co–evolutive Modelling

The proposed continuous co–evolutive model in Equation (15) may be solved by recognizing that it is a special version of Equation (19) (see Appendix A). In this sense we have to determine, preliminarily, the potential market $m(t)$ on the basis of Equation (9) and Equation (19). For the initial conditions $m(0)=0$, $f(\cdot)=1$ and

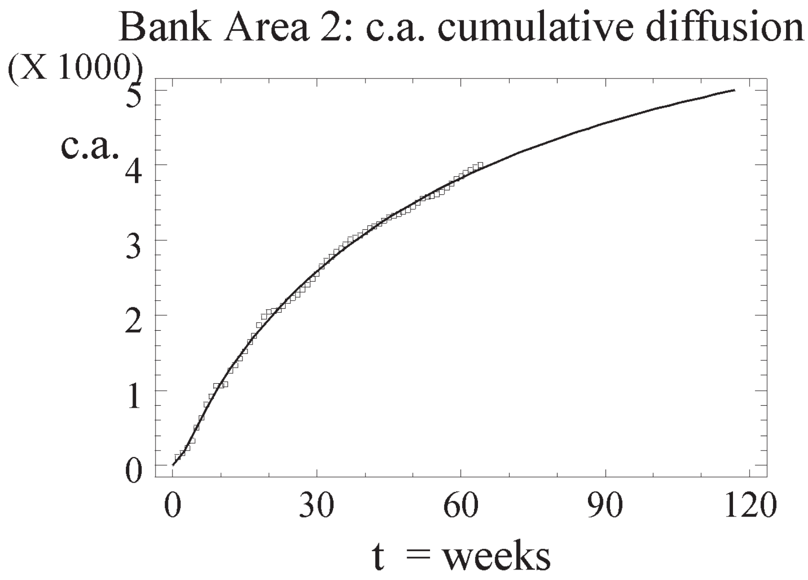

Figure 2: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule.

$g(\cdot) = 1$, we obtain

$$m(t) = K\sqrt{\frac{1 - e^{-D_c t}}{\frac{1}{{}_c r_2} - \frac{1}{{}_c r_1} e^{-D_c t}}}, \quad D_c = \sqrt{(q_c - p_c - e_c)^2 + 4(q_c + w_c)p_c} > 0, \tag{16}$$

where ${}_c r_i = (-(q_c - p_c - e_c) \pm D_c)/(-2(q_c + w_c)),\ i = 1, 2$, with ${}_c r_2 > {}_c r_1$. If, for instance, $e_c > 0$ then the limit of $m(t)$ for $t \to +\infty$ may be less than K.

Vice versa, note that if communication effects are *persistent*, i.e. with no decay effect, $e_c = 0$, and no negative word–of–mouth, $w_c = 0$, then $D_c = q_c + p_c$ and ${}_c r_1 = -p_c/q_c,\ {}_c r_2 = 1$ so that

$$m(t) = K\sqrt{\frac{1 - e^{-(p_c + q_c)t}}{1 + \frac{q_c}{p_c} e^{-(p_c + q_c)t}}}. \tag{17}$$

The limiting behaviour of $m(t)$ for $t \to +\infty$ equals the constant carrying capacity K.

Under an initial condition $C = 0$, for $g(\cdot) = m(\cdot)$ and $f(\cdot) = x(\cdot)$ the perturbed co–evolutive model, controlled by Equation (15) is determined on the basis of Equation (19) (see Appendix A),

$$y(t) = m(t)\frac{1 - e^{-D_s \int_0^t x(\tau)d\tau}}{\frac{1}{{}_s r_2} - \frac{1}{{}_s r_1} e^{-D_s \int_0^t x(\tau)d\tau}}, \quad D_s = \sqrt{(q_s - p_s - r_s)^2 + 4q_s p_s} > 0, \tag{18}$$

where ${}_sr_i = (-(q_s - p_s - r_s) \pm D_s)/(-2q_s),\ i = 1, 2$, with ${}_sr_2 > {}_sr_1$.

The perturbed closed form solution is very useful for a statistical approach to forecasting and simulations. The internal rules generating a two–fold NA under a widespread distribution of local influence on individual adoption or withdrawal of an innovation are represented by the involved parameters that may be used, within stochastic rules (7) and (12), for simulations under scenario hypotheses.

The time dependent potential market $m(t)$ penalizes with different emphasis the evolution of the natural adoption process. In Figure 1 we represent two different communication frameworks. In case (a) we consider a good positive w–o–m, $q_c = 0.7$, and an absent effect of negative w–o–m, $w_c = 0$. In case (b) we have considered a negative w–o–m, 0.2, which is not compensated by a stronger positive imitative component, $q_c = 0.9$. Case (b) exhibits a lower asymptotic potential market.

The statistical implementation of model (18) may adopt different error structures. In a nonlinear regressive approach we consider a particular model for observations, $w(t) = y(t) + \varepsilon(t)$, with an i.i.d. residual $\varepsilon(t)$. A useful complementary approach is based on ARMAX representation with a standard nonlinear estimation as a first step (see e.g. Guseo (2004), Guseo and Dalla Valle (2005) and Guseo et al. (2006)).

Note that joint identifiability of parameters in Equation (9) is not possible because the autonomous Riccati Equation (19), under $f(\cdot) = g(\cdot) = 1$, is characterized by three independent parameters so that we have to evaluate which are the dominant effects or, more generally, we have to set one of the four parameters in Equation (9) to a specified level based upon past experience. A common choice is e_c or w_c exclusion.

5 A Current Account Diffusion

We examine the weekly cumulative diffusion of a particular bank current account introduced by Cardine in a northern area of Italy (Area 2) for small and medium size firms. The cumulative data refer to a 64 weeks period from the origin of the service. Original data inspection suggests us that the exit rules parameters at both levels (communication network and adoption process) may be considered, at a first step, non significant, i.e., $e_c = w_c = r_s = 0$.

Following these assumptions we implement our model (18) with variable potential in order to understand its performance under a nonlinear regressive framework. The main results are outlined in Table 1.

We observe a quite interesting determination index, $R^2 = 0.998825$, which is confirmed by a good graphical performance, see Figure 2. Nevertheless, the Durbin-Watson statistic (0.444564) suggests the presence of autocorrelated residuals. Note that residual deviance is SSE = 135785 and local deviations in the first part of non cumulative series is very high (see, for instance, data description in Figure 3).

Under such conditions the marginal linearized asymptotic 95% confidence intervals are instable so that we may exclude their marginal direct use. Nevertheless, global use of transfer function is unaffected. We argue that this problem may be

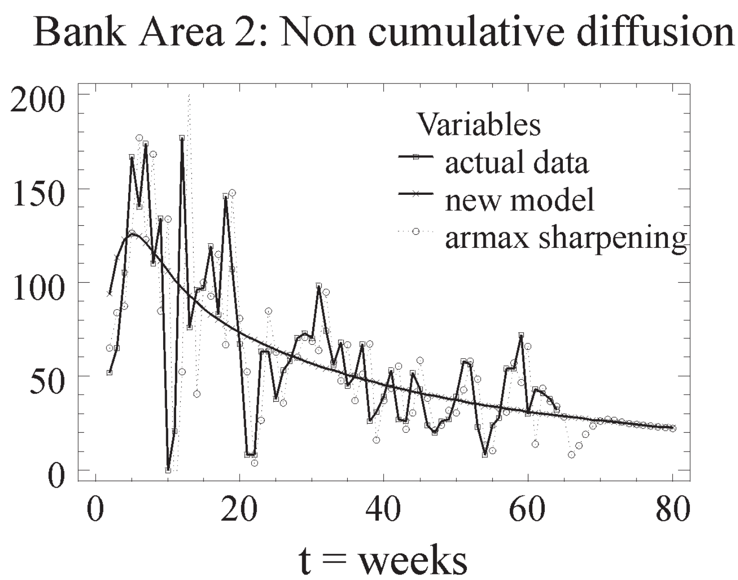

Figure 3: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive non cumulative model with no exit rule, ARMAX sharpening, standard instantaneous Bass model and actual "active bank account" data.

Table 1: Current account diffusion. Parameters estimates of co–evolutive model for Cardine Area 2 data with no exit rule. ( ) marginal linearized asymptotic 95% confidence limits

K	q_c	p_c	q_s	p_s	R_1^2	$D-W$
6883.1	0.1840	0.1730	-0.0164	0.0192	0.998265	0.444
(-9508)	(-0.357)	(-0.030)	(-0.0721)	(-0.0252)	SSE :	
(23274)	(0.725)	(0.376)	(0.0394)	(0.0636)	[135785]	

overcome by implementing an appropriate ARMAX procedure. The main results are outlined in Table 2.

The proposed ARMAX procedure considers only an AR component of order two with a regressor based upon the predicted values of initial NLS step referred to the new co–evolutive model with variable potential (PREb2cobs000). The goodness–of–fit is very high, $R_2^2 = 0.999427$, (see, for instance, Figure 4 and Figure 3). In particular, in Figure 3 we compare the non cumulative diffusion bank account data and competing models. Note the dominant performance of new composed model and the perfect agreement of NLS-ARMAX representation with reference to current data.

The residual deviance is one third of previous one: $SSE = 45616 = 747.8 \cdot 61$ so that the squared multiple partial correlation coefficient is $\tilde{R}^2 = 0.664$ and

Table 2: Co-evolutive cumulative model with no exit rule and ARMAX(2,0,0) sharpening. ( ) t–statistic; [ ] p–values

$AR(1)$	$AR(2)$	$PREb2cobs000$	$mean$	SSE
1.0729	-0.3886	0.3080	118.538	45616
(9.422)	(-4.569)	(5.4987)	(2.8222)	$\{d.f.61\}$
[0.000000]	[0.000025]	[0.000001]	[0.006430]	$R_2^2 = 0.999427$

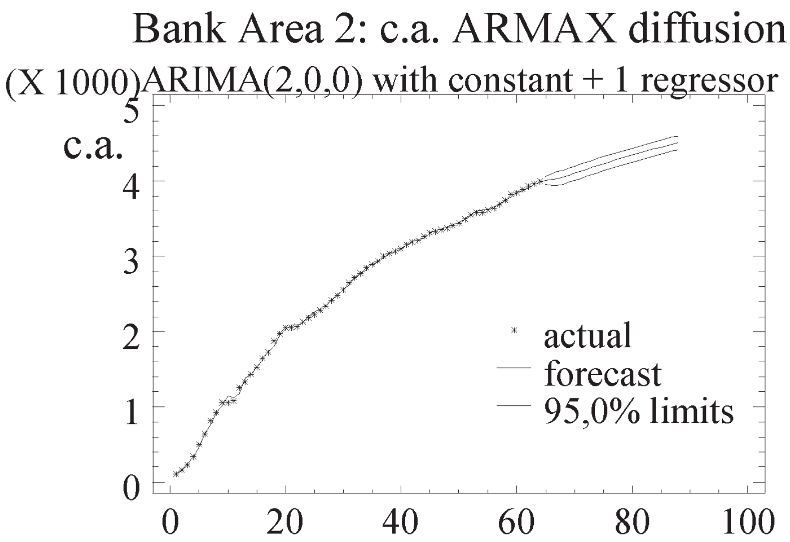

Figure 4: Current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule and cumulative ARMAX sharpening.

the corresponding $F = \tilde{R}^2(N-k)/((1-\tilde{R}^2)s)$ ratio – where s is the incremental parameter number between nested models and k is the parameter cardinality of extended ARMAX model – is quite significant, $F = 36.9$. Previous first step based on non linear variable potential model (PREb2cobs000) and AR components are marginally significant.

Following these results, evaluation of potential market (17) – which is essentially a latent structure that we can not measure directly – may be compared with the approximate averaged dynamics described by model (18). As we can see, by inspecting Figure 5, the inferred potential market reaches its stationary level after ten weeks demonstrating that the joint communication and marketing effort effects are very rapid. Both parameters p_c and q_c are quite high: 0.173 and, respectively, 0.184, with an expected high value for p_c that represents the direct bank communication effort.

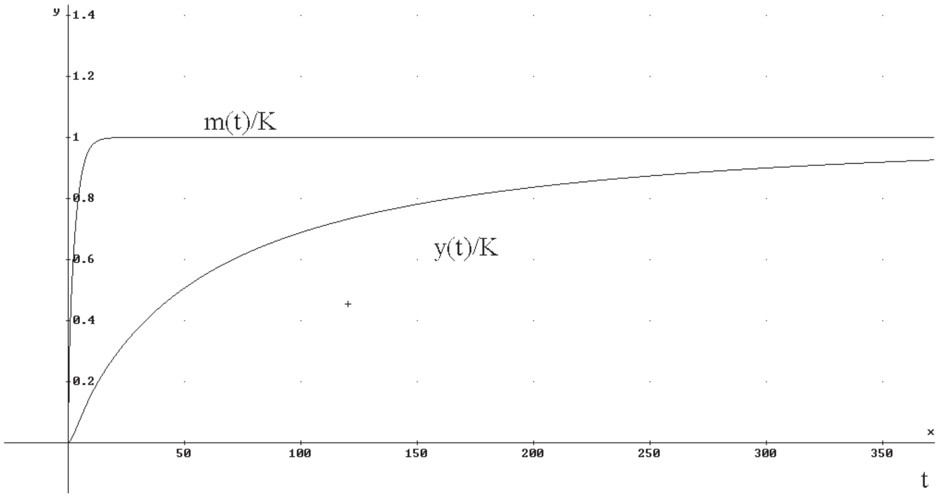

Figure 5: Normalized current account diffusion (Area 2, Cardine, Italy). Co–evolutive cumulative model with no exit rule and normalized potential market.

6 Final remarks and discussion

This paper addresses different aspects in innovation diffusion modelling by combining theoretical, technical and applied aspects on communication dynamics and adoption processes. Here we summarize some crucial elements we have highlighted:

a) Innovation diffusion is not a univariate adoption process over time. We argue that an adoption process is nested in a communication network that evolves dynamically and implicitly generates the corresponding non–constant potential market.

b) We guess that a communication network is a necessary phase in determining the evolution of a *prior related knowledge*, which is, using Cohen and Levinthal's (1990) terminology, the basic element for developing an *absorptive capacity.*

c) Our two-phase modelling is a particular specification of the above general ideas. Indeed, some opportunities and problems may be better examined within a well-defined mathematical and statistical framework in order to test performances, significance of model components and forecasting.

d) Cellular Automata and Network Automata are a simple and effective tool for representing both the communication network evolution and the nested adoption process.

e) In our model we assume that the communication network is not observable. In general we do not have precise information about how agents communicate between them and the network we consider has a virtual structure. However we are not interested in determining detailed particular shapes of the actual network. Our focus is on an aggregate transformation of this network, i.e., the concrete potential market $m(t)$.

f) With a *mean field approximation* we have reformulated a Complex Systems representation in a dual tractable differential one, see Equations (9), (11) and (15).

g) In our model, especially with reference to the Riccati Equation (15), functions $m(t)$, and $x(t)$ are independent tools. The effect of intervention function $x(t)$ modifies the time path of diffusion by locally expanding or shrinking adoptions within a "balance equation constraint". Instead, the potential market, $m(t)$, controls and modifies the size of this process, expressed in terms of the absolute amount of adoptions. This is a technical specification, useful to avoid theoretical misunderstandings between these different and separable effects.

h) The proposed application gives some insights on the role of statistics in analyzing evolving time series within a life cycle context. In particular, we observe that in this specific application the Mean Field Approximation, that allows an interesting aggregate description of a Complex Systems representation, does not consider effects of a supposed (not observed) heterogeneity of adopters (or adoptions). Nevertheless, an ARMAX sharpening, applied as a second step after a nonlinear least squares procedure, completes inference in a satisfactory way.

i) The substantive implication of our model, is that we are able to estimate, in an indirect way and under appropriate theoretical assumptions, the character of an evolving potential market simply using cumulative selling data. This is of particular concrete interest because it allows to measure indirectly the receptiveness of a social context, facilitating comparisons between different situations and evaluations on the effectiveness of firms' marketing efforts.

7 APPENDIX A: Riccati Equation, a Special Case

Let us consider the following special Riccati equation in (X,Y) real space

$$y'_x = a\frac{f(x)}{g(x)}y^2 + \left(bf(x) + \frac{g'(x)}{g(x)}\right)y + cf(x)g(x), \tag{19}$$

where $a, b, c \in R$, $D = \sqrt{b^2 - 4ac} > 0$ and $g(x) \neq 0$, $f(x)$ are real functions.

We note that this special version of non autonomous Riccati equation is not examined in the well–known *Handbook* by Polyanin and Zaitsev (2003).

The analysis proposed in the sequel represents a contribution to the Polyanin's Cathaloge.

An equivalent form of Equation (19) is

$$\frac{y'_x g(x) - g'(x)y}{g(x)} = \left(\frac{a}{g(x)}y^2 + by + cg(x)\right)f(x), \tag{20}$$

or

$$\frac{y'_x g(x) - g'(x)y}{g^2(x)} = \left[a\left(\frac{y}{g(x)}\right)^2 + b\left(\frac{y}{g(x)}\right) + c\right]f(x). \tag{21}$$

With a simple substitution, i.e., $z = y/g(x)$, we have

$$z' = (az^2 + bz + c)f(x) \tag{22}$$

for which a general solution is attainable.

Let us consider the real roots of equation $az^2+bz+c = 0$, i.e., $r_i = (-b \pm D)/2a \in R, i = 1, 2$, where $D = a(r_2 - r_1) = \sqrt{b^2 - 4ac} > 0$ so that Equation (22) may be represented as follows

$$z' = a(z - r_1)(z - r_2)f(x). \tag{23}$$

Consider the substitution $\dot{z} = z - r_2$ with $\dot{z}' = z'$ and initial conditions $z(0) = C$ or $\dot{z}(0) = C - r_2$ then, dividing both member of transformed previous equation by $\dot{z}^2$, we attain $\frac{\dot{z}'}{\dot{z}^2} = a(\dot{z} + r_2 - r_1)\frac{1}{\dot{z}}f(x)$, or $\frac{\dot{z}'}{\dot{z}^2} = \left\{a(r_2 - r_1)\frac{1}{\dot{z}} + a\right\} f(x)$.

Let us consider a further substitution, i.e., $\hat{z} = \frac{1}{\dot{z}}$, with $\hat{z}' = -\frac{\dot{z}'}{\dot{z}^2}$ and initial condition $\hat{z}(0) = \frac{1}{C-r_2}$ so that we obtain equation

$$-\hat{z}' = \{a(r_2 - r_1)\hat{z} + a\}f(x)\,, \tag{24}$$

which may be integrated as a linear first order equation (see, e.g. Apostol (1978, p. 31)). Its solution is

$$\hat{z} = \frac{1}{C - r_2}G(x) + G(x)a\int_0^x f(\tau)e^{-a(r_2-r_1)\int_0^\tau f(\xi)d\xi}d\tau, \tag{25}$$

where $G(x) = e^{a(r_2-r_1)\int_0^x f(\tau)d\tau}$ or equivalently $G(x) = e^{D\int_0^x f(\tau)d\tau}$ so that

$$\begin{aligned}\hat{z} &= \frac{1}{C - r_2}G(x) + G(x)a\left[-\frac{1}{D}e^{-D\int_0^x f(\xi)d\xi} + \frac{1}{D}\right]\\ &= \frac{G(x)}{C - r_2} - \frac{1}{(r_2 - r_1)}\left[1 - G(x)\right] = \frac{r_2 - r_1G(x) - C(1 - G(x))}{(C - r_2)(r_2 - r_1)}.\end{aligned} \tag{26}$$

Let us express solution (26) in terms of the initial variable, $z = \frac{1}{\hat{z}} + r_2$,

$$\begin{aligned}z &= r_2 + \frac{(C - r_2)(r_2 - r_1)}{r_2 - r_1G(x) - C(1 - G(x))}\\ &= \frac{r_1r_2(1 - G(x)) - C(r_1 - r_2G(x))}{r_2 - r_1G(x) - C(1 - G(x))}.\end{aligned} \tag{27}$$

We obtain the general solution of Equation (19) in a straightforward manner, i.e.,

$$y(x) = g(x)\frac{r_1r_2(1 - G(x)) - C(r_1 - r_2G(x))}{r_2 - r_1G(x) - C(1 - G(x))}. \tag{28}$$

If the initial condition is set to zero, $C = 0$, we obtain,

$$y(x) = g(x)\frac{1 - G^{-1}(x)}{\frac{1}{r_2} - \frac{1}{r_1}G^{-1}(x)} = g(x)\frac{1 - e^{-D\int_0^x f(\tau)d\tau}}{\frac{1}{r_2} - \frac{1}{r_1}e^{-D\int_0^x f(\tau)d\tau}}. \tag{29}$$

If $\lim_{x\to\infty}\int_0^x f(\tau)d\tau = +\infty$, we attain an interesting limiting behaviour of $y(x)$, i.e., $\lim_{x\to\infty} y(x) = r_2 \lim_{x\to\infty} g(x)$.

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