Estimation for ratio of two population means in double sampling

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Abstract

Using two-phase sampling scheme, a general class of estimators for ratio and product of two finite population means is proposed. This class depends on the sample means and variances of two auxiliary variables. The minimum variance bound for any estimator in the class is provided (up to terms of order \(n^{-1}\)). It is shown that there exists at least a chain regression type estimator which reaches this minimum. A numerical illustration is also given.

keywords: two-phase sampling – auxiliary variable – regression type estimator.

1 Introduction

In many practical settings the estimation of ratio and product of two population means is of interest. An example is the ratio "weight/height" for determining the body mass index. The use of an auxiliary variable in this estimation problem has been widely discussed. When the population mean of the auxiliary variable is unknown a double sampling scheme may be used. In this context, Khare (1991) proposes a class of estimators based on only one auxiliary variable, while Chand (1975) gets a chain ratio type estimator, replacing the unknown population mean of the auxiliary variable with an estimate based on a second auxiliary variable with known population mean. Other authors followed this idea, as Srivastava et al. (1988,1989), Prasad et al. (1996) and Singh et al. (1994). On the other hand, Singh and Singh (1994) consider only one auxiliary variable but, as an additional information to improve the estimate of the population ratio, they use both sample means and variances of the auxiliary variable. In this paper we generalize this idea, using the first and second phase sample means and variances of two auxiliary variables.

In the second Section we describe a general class of estimators which include all the previously quoted estimators. Furthermore we provide the minimum attainable MSE for these estimators, up to terms of \(O(n^{-1})\). We will call it the minimum variance bound, since MSE and variance are the same at the first order of approximation. We finally provide a regression type estimator which reach this minimum and we call it the "best" estimator in the class. Actually, any other estimator which is equivalent to the best estimator at the first order of approximation is optimum as well. Some examples of optimum estimators will be provided in Section 3. In that section it is also shown that adding auxiliary information leads to more efficient estimators. This result is also confirmed by an empirical study in Section 4.

2 A general class of estimators

Let \(U = \{1, \ldots, i, \ldots, N\}\) be a finite population and \(U_i\) be the value of an observable variable \(U\), for the \(i\)-th population unit. Then, the population mean and variance of \(U\)
are \( U = \sum_{i=1}^{N} U_i / N \) and \( S_0^2 = V(U) = \sum_{i=1}^{N} (U_i - U)^2 / (N - 1) \).

Denoting the study variables by \( Y_j, j = 0, 1 \), we are interested in estimating the population ratio \( R = \bar{Y}_0 / \bar{Y}_1 \), when sample information about two auxiliary variables, \( X \) and \( Z \), are available. Specifically, we assume that \( X \) and \( Z \) are related with \( Y_j, j = 0, 1 \), but there is not any information about the population mean and variance of \( X \). In that case estimation of \( R \) can be based on double sampling. Thus, we assume that a preliminary large sample of \( n' \) \((n' < N)\) units is drawn by a simple random sample without replacement (SRSWOR). At this phase only \( X \) and \( Z \) are measured. As a second step a smaller sample of size \( n \) \((n < n')\) is drawn from the first phase sample, by a SRSWOR as well. Without loss of generality, at this phase all the variables \( Y_j, j = 0, 1, X \) and \( Z \) are observed.

Denoting the first and second phase sample means and variances of any variable \( U \), by \( \bar{u}, s_u^2 \) and \( \bar{u}, s_u^2 \), the proposed class of estimators is defined as

\[
R_g = g \left( \hat{R}, t^T \right),
\]

where \( t = (s_X^2, \bar{X}, s_Z^2, \bar{Z}, s_{\bar{X}}^2, \bar{\bar{X}})^T \), \( \hat{R} = \bar{Y}_0 / \bar{Y}_1 \) and \( g \) is a function such that

a. \( g : S \to \mathbb{R} \) where \( S = \mathbb{R}^8 \) is a convex and bounded set which contains the point \( (R, T^T)^T \), where \( T = E(t) = (S_X^2, \bar{X}, S_Z^2, \bar{Z}, S_{\bar{X}}^2, \bar{\bar{X}})^T \).

b. It is a continuous and bounded function in \( S \).

c. Its first and second partial derivatives are continuous and bounded in \( S \).

d. \( g(\hat{R}, T^T) = \hat{R} \).

Let

\[
g_0 = \frac{\partial g(\hat{R}, t^T)}{\partial \hat{R}} \bigg|_{(\hat{R}, t^T) = (R, T^T)}, \quad \text{and} \quad g_i = \frac{\partial g(\hat{R}, t^T)}{\partial t_i} \bigg|_{(\hat{R}, t^T) = (R, T^T)}, \quad i = 1, \ldots, 8
\]

be the partial derivatives of \( g \) with respect to the first component, \( \hat{R} \), and the other components, \( t_i, i = 1, \ldots, 8 \), respectively. From point (d), we have that \( g(R, T^T) = R \) and \( g_0 = 1 \).

Expanding \( R_g \) at the point \( (R, T^T) \) in a second order Taylor's series we have

\[
R_g \cong \hat{R} + (s_X^2 - S_X^2) g_1 + (\bar{X} - \bar{X}) g_2 + (s_Z^2 - S_Z^2) g_3 + (\bar{Z} - \bar{Z}) g_4 + (s_{\bar{X}}^2 - S_{\bar{X}}^2) g_5 + (\bar{\bar{X}} - \bar{\bar{X}}) g_6 + (s_{\bar{X}}^2 - S_{\bar{X}}^2) g_7 + (\bar{\bar{X}} - \bar{\bar{X}}) g_8.
\]

Since the population mean and variance of \( X \) are unknown, we have to impose the following constraints, \( g_7 = -g_1 \) and \( g_8 = -g_2 \). Furthermore, expanding \( \hat{R} \) at the point \( (\bar{Y}_0, \bar{Y}_1) \) in a second order Taylor's series, we have that

\[
\hat{R} \cong R \left( 1 + \frac{\bar{Y}_0}{\bar{Y}_0} - \frac{\bar{Y}_1}{\bar{Y}_1} \right) = R (1 + d),
\]
where $\tilde{\delta}$ is the second phase sample mean of $D = Y_0 / Y_0 - Y_1 / Y_1$. For computational reasons, it is convenient to define the random vector $\tilde{\mathbf{v}} = (\tilde{X}^T, \tilde{Z}^T)^T$ where, for any variable $U$, $\tilde{U} = (\delta_U, U)^T$ and $\tilde{\delta}_U = (U - \bar{U})^T$.

Thus, equation (2) can be rewritten as

$$R_g \cong \tilde{R} + (t_\tilde{\mathbf{v}} - t_\mathbf{v})^T g_{\mathbf{v}} + (t_2' - T_2) g_2$$

$$\cong R(1 + \tilde{\alpha}) + (t_\tilde{\mathbf{v}} - t_\mathbf{v})^T g_{\mathbf{v}} + (t_2' - T_2) g_2$$

where $t_2'$ and $t_\tilde{\mathbf{v}}$ are the first and second phase sample estimates of $T_\mathbf{v} = (S_X^2, \overline{X}, S_Z^2, \overline{Z})^T$,

$t_\mathbf{v}$ is the first phase sample estimate of $T_\mathbf{v} = (S_X^2, \overline{Z})^T$, $g_\mathbf{v} = (g_1, g_2, g_3, g_4)^T$ and $g_2 = (g_5, g_6)^T = (g_5 + g_6, g_6 + g_4)^T$.

From now on, the quantities involving $Y_0$ and/or $Y_1$ will be identified by $0$ and/or $1$ respectively, e.g. $\rho_{0,1}$ will denote the correlation coefficient between $Y_0$ and $Y_1$. Then the first order approximation for $\text{MSE}(R_g)$ is given by

$$\text{MSE}(R_g) \cong R^2 \theta_2 S_D^2 + \theta g_{\mathbf{v}}^T S_{\mathbf{v}\mathbf{v}} g_{\mathbf{v}} + \theta_1 g_2^T S_{22} g_2 + 2 R \left( \theta g_{\mathbf{v}}^T S_{D\mathbf{v}} + \theta_1 g_2^T S_{D2} \right),$$

where

$$\theta_1 = \frac{1}{n'} - \frac{1}{N}, \quad \theta_2 = \frac{1}{n} - \frac{1}{N}, \quad \theta = \frac{1}{n} - \frac{1}{n'}$$

$$S_D^2 = C_0^2 + C_1^2 - 2 \rho_{0,1} C_0 C_1, \quad C_j = S_j / \overline{Y}_j, \quad j = 0, 1$$

$$S_{D\mathbf{v}} = \begin{bmatrix} S_{\mathbf{v}\mathbf{v}} \\ S_{D2} \end{bmatrix} = \begin{bmatrix} \text{Cov}(D, \delta_X) \\ \text{Cov}(D, X) \\ \text{Cov}(D, \delta_Z) \\ \text{Cov}(D, Z) \end{bmatrix}$$

and

$$S_{\mathbf{v}\mathbf{v}} = \begin{bmatrix} S_{\mathbf{v}\mathbf{v}} & S_{\mathbf{v}2} \\ S_{\mathbf{v}2} & S_{22} \end{bmatrix}$$

with

$$S_{\mathbf{v}\mathbf{v}} = \begin{bmatrix} V(\delta_X) & \text{Cov}(\delta_X, X) \\ \text{Cov}(\delta_X, X) & V(X) \end{bmatrix}, \quad S_{22} = \begin{bmatrix} V(\delta_Z) & \text{Cov}(\delta_Z, Z) \\ \text{Cov}(\delta_Z, Z) & V(Z) \end{bmatrix},$$

$$S_{\mathbf{v}2} = \begin{bmatrix} \text{Cov}(\delta_X, \delta_Z) & \text{Cov}(\delta_X, Z) \\ \text{Cov}(\delta_X, \delta_Z) & \text{Cov}(X, Z) \end{bmatrix}, \quad \text{and} \quad S_{22} = S_{\mathbf{v}2}^T.$$
and replacing them in (4) we get the minimum first order approximation of \( \text{MSE}^*(\cdot) \), denoted by \( \text{MSE}^*(\cdot) \). It is the minimum variance bound for all the estimators based on the auxiliary vector \( t \) and has a simple form,

\[
\text{MSE}^*(R_g) = R^2 S_D^2 \left[ \theta_1 \left( 1 - \rho_{D,Z}^2 \right) + \theta \left( 1 - \rho_{D,V}^2 \right) \right]
\]

\[
= \text{MSE}'(\hat{R}) - R^2 S_D^2 \left[ \theta_1 \rho_{D,Z}^2 + \theta \rho_{D,V}^2 \right],
\]

where \( \text{MSE}'(\hat{R}) = \theta_2 R^2 S_D^2 \) is the first order approximation of \( \text{MSE}(\hat{R}) \) and

\[
\rho_{D,U}^2 = \frac{S_{D,U}^T S_{D,U}^{-1} S_{D,U}}{S_D^2}
\]

denotes the multiple correlation coefficient between \( D \) and any one random vector \( U \).

Actually, \( g_V^* \) and \( g_Z^* \) are not known, i.e., they depend on the population ratio \( R \) and on the usually unknown regression coefficients of \( D \) on \( V \) and \( Z \). Anyway, if approximate unbiased estimates, \( \hat{g}_V^* \) and \( \hat{g}_Z^* \), are available, we can define the following class

\[
\hat{R}_g = g(\hat{R}, t^T, \hat{g}^* T)
\]

which is equivalent to class (1), at the first order of approximation. Thus, replacing \( \hat{g}_V^* \) and \( \hat{g}_Z^* \) in equation (3) we get a chain regression type estimator which reaches \( \text{MSE}^*(R_g) \).

Let us denote it as \( \hat{R}_{reg} \). For giving the expression of \( \hat{R}_{reg} \) we need more notation. Let \( U_1 \) and \( U_2 \) be any two auxiliary variables (where \( U_2 \) can be a random vector) and let us denote the partial regression coefficient of \( Y_1 \) on \( U_1 \) given \( U_2 \) as \( \beta_{j,1|2} \), \( j = 0, 1 \). If \( \hat{\beta}_{D,U_1|U_2} \) is an estimate of \( \beta_{D,U_1|U_2} = \beta_{0,U_1|U_2}/\bar{V}_0 - \beta_{1,U_1|U_2}/\bar{V}_1 \), then \( \hat{R}_{reg} \) is given by

\[
\hat{R}_{reg} = \hat{R} \left[ 1 - \hat{\beta}_{D,\delta x|\bar{x},\bar{z}} (s_X^2 - s_X^2) - \hat{\beta}_{D,\delta x|\bar{x},\bar{z}} (\bar{x} - \bar{x}) - \hat{\beta}_{D,\delta x|\bar{x},\bar{z}} (s_Z^2 - s_Z^2) \right]
\]

\[
- \hat{\beta}_{D,\delta z|\bar{z}} (\bar{z} - \bar{z}) - \hat{\beta}_{D,\delta z|\bar{z}} (s_Z^2 - s_Z^2) - \hat{\beta}_{D,\delta z|\bar{z}} (\bar{x} - \bar{x}) \right].
\]

We call \( \hat{R}_{reg} \) as the “best” estimator in the class \( \hat{R}_g \) but any other estimator which is at the first order equivalent to \( \hat{R}_{reg} \) is optimum as well.

**Remark.** If the interest is in the population product \( P = \bar{Y}_0 \bar{Y}_1 \) all the previous results are still valid. It is enough to replace \( R \) and \( \hat{R} \) with \( P \) and \( \hat{P} \) respectively, where now \( \hat{P} = \bar{y}_0 \bar{y}_1 \equiv P(d - 1) \) with \( D = Y_0/\bar{Y}_0 + Y_1/\bar{Y}_1 \).

### 3 A partial use of the auxiliary variables

In the previous section the whole auxiliary vector \( t = (s_X^2, \bar{x}, s_Y^2, \bar{z}, s_X^2, \bar{z}, s_Y^2, \bar{x})^T \) was assumed to be known. When only some of these auxiliary quantities are available (or some informations are ignored) all the results of Section 2 can be easily adapted. In Table I some classes of estimators which are based on different vectors \( t \) are listed.
<table>
<thead>
<tr>
<th>Vector $t$</th>
<th>Proposed classes of (or) estimators</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_4 = (\overline{x}, \overline{x}', \overline{z}, \overline{z}')^T$</td>
<td>$\hat{R}_A = f\left(\hat{R}, \frac{\overline{x}}{\overline{z}}, \frac{\overline{x}'}{\overline{z}'}\right)$</td>
<td>Ahmed (1997)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}<em>{SS} = \frac{\mu - \hat{R}</em>{Ax}(\overline{x} - \overline{z})(\overline{x}' - \overline{z}')} {\mu - \hat{R}_{Ax}(\overline{x} - \overline{z})(\overline{x} - \overline{z}')}$</td>
<td>Singh and Sigh (1997)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}_{SRKS} = \hat{R} \left(\frac{\overline{x}}{\overline{z}}\right)^{\alpha_1} \left(\frac{\overline{x}'}{\overline{z}'}\right)^{\alpha_2}$</td>
<td>Srivastava et al. (1988)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}_S = f\left(\hat{R}, \frac{\overline{x}}{\overline{z}}, \frac{\overline{x}'}{\overline{z}'}\right)$</td>
<td>Singh et al. (1994)</td>
</tr>
<tr>
<td>$t_3 = (\overline{x}, \overline{x}', \overline{z})^T$</td>
<td>$\hat{R}_{PPS} = \hat{R} \frac{\overline{x} + \alpha_1 (\overline{x} - \overline{z})}{\overline{x} - \alpha_2 (\overline{x} - \overline{z})}$</td>
<td>Prasad et al. (1996)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}<em>{SS} = \frac{\mu - \hat{R}</em>{Ax}(\overline{x} - \overline{z})(\overline{x} - \overline{z})}{\mu - \hat{R}<em>{Ax}(\overline{x} - \overline{z})(\overline{x} - \overline{z})}$, $\hat{\lambda} = \frac{\overline{z}(\hat{R}</em>{Ax} - \hat{R}<em>{Ax})}{\overline{z}(\hat{R}</em>{Ax} - \hat{R}_{Ax})}$</td>
<td>Singh and Sigh (1997)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}_{K} = \hat{R} \left[\alpha_0 \left(\frac{\overline{x}}{\overline{z}}\right)^{\alpha_1} + (1 - \alpha_0) \left(\frac{\overline{x}'}{\overline{z}'}\right)^{\alpha_2}\right]$</td>
<td>Khare and Srivastava (1998)</td>
</tr>
<tr>
<td>$t_2 = (\overline{x}, \overline{x}')^T$</td>
<td>$\hat{R}_{K} = \hat{R} \left(\frac{\overline{x}}{\overline{z}}\right)$</td>
<td>Khare (1991)</td>
</tr>
<tr>
<td></td>
<td>$\hat{R}<em>{SS} = \frac{\mu + \hat{R}</em>{Ax}(\overline{x} - \overline{z})}{\mu + \hat{R}_{Ax}(\overline{x} - \overline{z})}$</td>
<td>Singh and Sigh (1997)</td>
</tr>
</tbody>
</table>

These classes or estimators were proposed following different approaches but they belong to the specific class $R_\omega$ defined by the chosen vector $t$. Furthermore, they are optimal in that class, even if this is not a general rule. For instance, any estimator in the following class (Prasad et al., 1996)

$$R_\omega = \hat{R} \left(\frac{\overline{x} + (1 - \omega) \overline{z}}{\overline{z}}\right)$$

cannot be optimal in the general class defined by vector $t_3 = (\overline{x}, \overline{x}', \overline{z})^T$, since it is not equivalent to the best estimator. As a matter of fact, the second order Taylor’s expansion of $R_\omega$,

$$R_\omega = \hat{R} + R \frac{\omega}{\overline{X}} (\overline{x} - \overline{x}') + R \frac{1 - \omega}{\overline{Z}} (\overline{z} - \overline{z})$$

depends on only one unknown parameter while equation (3) depends on two parameters and thus minimizing the corresponding MSE*(')’s, with respect to the unknown parameters, a lowest value will be got in the latter case.
Choosing different auxiliary quantities we may have several general classes of estimators. However, the best choice is the use of the whole vector $t$ since

$$\text{MSE}^*(\hat{R}_2) \geq \text{MSE}^*(\hat{R}_3) \geq \ldots \geq \text{MSE}^*(\hat{R}_k),$$

where $\hat{R}_2$ is $\hat{R}_g$ based on the simplest auxiliary vector $t_2 = (\overline{x}, \overline{x}')^T$, $\hat{R}_3$ is based on $t_3 = (\overline{x}, \overline{x}', \overline{x}'')^T$ and so on up to $\hat{R}_k$ where the whole vector of auxiliary quantities is considered. This result follows from that $\text{MSE}^*(R_g)$ depends on two multiple correlation coefficients, which satisfy the following inequality

$$\rho_{D, U}^2 \geq \rho_{D, U_-}^2,$$

where $U = (U_1^T, U_k)^T$ is a $k \times 1$ random vector and $U_-$ is the $(k-1) \times 1$ vector got removing the last variable $U_k$ from $U$ (for the details see Diana and Tommasi (2002)).

4 Numerical illustration

In order to exhibit the performance of the best estimator when we use different auxiliary vectors $t$ we give here a numerical illustration. The data under consideration consist of 2247 woman patients of osteoporosis disease. The weight and height of these women are measured for estimating the corresponding ratio, that is the body mass index. Treating weight and height as variables $Y_0$ and $Y_1$, respectively, the number of pregnancies as $X$ and the age as $Z$, the following population values are obtained,

$$\overline{Y}_0 = 63.07, \quad \overline{Y}_1 = 161.01, \quad C_0 = 0.158, \quad C_1 = 0.037, \quad \rho_{01} = 0.424,$$

$$S_{\overline{Y}_0} = \begin{bmatrix} 6 & .54 & 18 & .54 \\ .54 & 1.4 & -8.24 & .89 \\ 18 & -8.24 & 10541 & 131 \\ .54 & .89 & 131 & 69 & \end{bmatrix}, \quad S_{\overline{Y}_1} = \begin{bmatrix} .00015 \\ .0265 \\ -1.079 \\ .072 & \end{bmatrix}.$$

Taking a first phase sample of size $n' = 100$ and a second phase sample of $n = 50$ units, we got the following sample quantities,

$$\overline{y}_0 = 65.86, \quad \overline{y}_1 = 162.7, \quad \overline{x} = 2.04, \quad \overline{x}' = 2, \quad \overline{z} = 57.16, \quad \overline{z}' = 58.51,$$

$$s_X^2 = 1.37, \quad s_Z^2 = 65.1,$$

$$s_{\overline{Y}_0} = \begin{bmatrix} 2.35 & -0.48 & 9.87 & 0.67 \\ -0.48 & 0.43 & 15.45 & 1.36 \\ 9.87 & 15.45 & 2284.13 & 103.9 \\ 0.67 & 1.36 & 103.9 & 40.71 \end{bmatrix}, \quad s_{\overline{Y}_1} = \begin{bmatrix} -2.59 \\ -4.82 \\ -2.08 \end{bmatrix}, \quad s_{\overline{Y}_1} = \begin{bmatrix} -0.45 \\ -1.34 \\ 1.11 \end{bmatrix}.$$

Table II shows the values of the best estimator, the corresponding $\text{MSE}^*$ and the relative efficiency over the standard estimator $\hat{R}$, for different choices of vector $t$. 

TABLE II
The standard and the best estimators of $R$ with their MSE.

<table>
<thead>
<tr>
<th>Vector $t$</th>
<th>Estimator</th>
<th>MSE*</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_2 = (\overline{x}, \overline{z})^T$</td>
<td>$\hat{R}_2 = .4047$</td>
<td>$6365 \times 10^{-8}$</td>
<td>.9880</td>
</tr>
<tr>
<td>$t_3 = (\overline{x}, \overline{z}, \overline{z})^T$</td>
<td>$\hat{R}_3 = .4046$</td>
<td>$6354 \times 10^{-8}$</td>
<td>.9863</td>
</tr>
<tr>
<td>$t'_4 = (\overline{x}, \overline{z}, s_X^2, s_z^2)^T$</td>
<td>$\hat{R}'_4 = .4051$</td>
<td>$6363 \times 10^{-8}$</td>
<td>.9877</td>
</tr>
<tr>
<td>$t_4 = (\overline{x}, \overline{z}, \overline{z})^T$</td>
<td>$\hat{R}_4 = .4045$</td>
<td>$6347 \times 10^{-8}$</td>
<td>.9853</td>
</tr>
<tr>
<td>$t_6 = (\overline{x}, \overline{z}, \overline{z}, s_X^2, s_z^2)^T$</td>
<td>$\hat{R}_6 = .4051$</td>
<td>$6343 \times 10^{-8}$</td>
<td>.9846</td>
</tr>
<tr>
<td>$t_8 = (\overline{x}, \overline{z}, \overline{z}, s_X^2, s_Z^2, s_X^2, s_z^2)^T$</td>
<td>$\hat{R}_8 = .4023$</td>
<td>$6309 \times 10^{-8}$</td>
<td>.9794</td>
</tr>
</tbody>
</table>

From the third column of this table we have that

$$\text{MSE}^*(\hat{R}_2) \geq \text{MSE}^*(\hat{R}_3) \geq \text{MSE}^*(\hat{R}_4) \geq \text{MSE}^*(\hat{R}_6) \geq \text{MSE}^*(\hat{R}_8),$$

and

$$\text{MSE}^*(\hat{R}'_4) \geq \text{MSE}^*(\hat{R}_6) \geq \text{MSE}^*(\hat{R}_8),$$

which is consistent with the general result given at the end of the previous section. On the other hand, that result cannot be applied any more if the information of an auxiliary vector $t$ is not completely included in the competitor. This is the case if we compare $\hat{R}_3$ with $\hat{R}_5$ or $\hat{R}_4$ with $\hat{R}'_4$. For this data set, in both cases $\hat{R}'_4$ is worse.

Finally, we note that the poor relative efficiencies, shown in the last column of Table II, are due to the small population correlation coefficients between interest and auxiliary variables: $\rho_{0,x} = .15$, $\rho_{1,x} = .038$, $\rho_{0,z} = .037$ and $\rho_{1,z} = -.08$, respectively. A more evident improvement is expected when those correlation coefficients increase.

5 Appendix

Let us assume $N$ large enough so that $(N-1)/N \approx 1$. We can get expression (4), for the first order approximation of MSE($\hat{R}_9$), using the following expected values, which are valid up to terms of order $O(n^{-1})$,

$$E \left[ (t_\nu - t'_\nu) (t_\nu - t'_\nu)' \right] \approx \theta S_{\nu,\nu}; \quad E \left[ (t_2 - T'_2) (t_2 - T'_2)' \right] \approx \theta_1 S_{2,2};$$

$$E \left[ (\overline{d} - \overline{D}) (t_\nu - t'_\nu)' \right] \approx \theta S_{D,\nu}; \quad E \left[ (\overline{d} - \overline{D}) (t_2 - T'_2)' \right] \approx \theta_1 S_{D,2}.$$
References


