Linear models and advertising*

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Abstract. Nerlove-Arrow's model is a starting point for some practical and theoretical studies in marketing. Here we want to give our point of view on this growing and exciting field of research. First of all we present the Nerlove-Arrow's linear model of goodwill evolution under advertising investment. Then we provide a sketch of a variety of problems which are based on it, recalling the different mathematical tools which are needed to discuss, and possibly solve them. We present some key problems in the panorama of the optimal control and differential games applications to advertising and mention some relevant literature, dating after 1994.
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"What about focusing your research on optimal control applications to Economics and Management Science, which is a rather promising field?"
Giovanni Castellani, 1982, to the third author.

1 Introduction

We want to present the linear model of goodwill evolution under advertising investment, which is due to Nerlove and Arrow, and provide a sketch of a variety of problems which are based on it, recalling the different mathematical tools which are needed to discuss, and possibly solve them.

Two main models are at the basis of the literature on optimal control applications to avertising and they have been proposed in about the same period. The first model, dated 1957, is due to Vidale and Wolfe [58]: the authors aim at modelling the sales response to advertising and try to represent some characteristic behaviors as observed in real data. The second model, dated 1962, is due to Nerlove and Arrow [47]: here the authors assume that the demand of a product (hence its sale intensity) depends on a state variable, called *goodwill*,

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that represents the effects of a firm investment in advertising. The two review articles [52] and [23] present the general context of the optimal control models in advertising until 1994: there we see how the Nerlove-Arrow's one has become an important reference for the advertising and marketing research.

Little [42] provides a first critical analysis of the two models, in view of a practical use of them on real data. He pays a special attention to the discrete time setting and presents his *Brandaid* model as a generalization of both Vidale-Wolfe's and Nerlove-Arrow's ones. Today, those two models are cornerstone references for many discrete time models used by marketing practitioners.

We will present some key dynamical optimization problems, which are based on the Nerlove-Arrow's dynamics, in the panorama of the optimal control and differential games applications to advertising. We will mention some seminal papers on the subject, together with some relevant literature, dating after 1994. In fact, the last decade has provided some contributions in new research directions, which are worth exploring more deeply.

The paper is organized as follows. In Section 2, we introduce Nerlove-Arrow's model and discuss its fundamental modelling features. In Section 3, we consider some typical optimal control problems, either in infinite or finite horizon, where an advertising intensity is looked for to optimize a suitable firm objective. In Section 4, we focus on differential games. In Section 5, we address the practically important issue of market segmentation. Finally, in Section 6, we look at the problems and opportunities posed by a multiplicity of advertising tools.

2 Advertising, goodwill, and demand

2.1 A direct relation between advertising and demand

In their model, Vidale and Wolfe [58] observe two main facts concerning the relation between sales and advertising. Sales intensity decreases with time if no advertising is done and, if an adequate advertising effort is done over a time period, then sales intensity increases, but a saturation effect may emerge. Hence they suggest that the sale intensity $s(t)$ satisfies the differential equation

$$\dot{s}(t) \;=\; \rho u(t)[1 - s(t)/m] - \delta s(t)\,, \tag{1}$$

where $u(t)$ is the advertising intensity, and $\rho, m, \delta > 0$ are parameters. This is a linear differential equation in the state variable $s(t)$, where the term $-\delta s(t)$ represents the spontaneous decay of the state. The parameter m is the saturation threshold, an upper bound to the sales intensity: when $m - s(t)$ is small and positive, the advertising intensity $u(t)$ must be large to sustain the sale level.

2.2 A mediated relation between advertising and demand

Nerlove and Arrow [47] propose a model in which the effect of advertising on sales is mediated by the *goodwill* variable. The goodwill state variable represents the effects of the firm investment in advertising and it affects the demand of the product together with price and other external factors. Goodwill is therefore seen as a stock of productive capital: it is subject to depreciation, i.e. spontaneous decay proportional to its value, and, on the other hand, it is sustained by the investment flow controlled by the firm. Nerlove and Arrow focus on the most elementary differential equation which describes an investment phenomenon in a capital stock subject to depreciation, i.e. the linear equation

$$\begin{cases} \dot{x}(t) &= u(t) - \delta x(t)\,, \\ x(0) &= \alpha\,, \end{cases} \tag{2}$$

where $x(t)$ and $u(t)$ are the capital stock and the investment intensity at time t, $\delta > 0$ is a decay parameter, which represents the capital depreciation over time, and α is the known value of the capital at the initial time 0. In fact equation (2) represents the capital dynamics of the neoclassical aggregate growth model [55, p. 432].

The model (2) is an approximation of Vidale-Wolfe's model (1) for large values of the saturation threshold m. In fact, if we let $m \to +\infty$, then (1) is equivalent to

$$\begin{cases} s(t) &= \rho x(t)\,, \\ \dot{x}(t) &= u(t) - \delta x(t)\,, \end{cases} \tag{3}$$

which represents a demand (sale) intensity $s(t)$ directly proportional to goodwill $x(t)$, which is driven by the advertising intensity $u(t)$ according to the differential equation (2).

The equation (2) is simpler to discuss than equation (1). Moreover, the goodwill variable may assume different meanings, depending on the particular context: a first example is sales intensity (see [14]), which is the most natural interpretation in view of the observed connection between Nerlove-Arrow's and Vidale-Wolfe's models. A second one, provided by [40], is *reservation price*. A third one, used by [60] in the discrete time version of the model, is *awareness*.

In the following we consider a variety of problems which are based on the equation (2) and on the Nerlove-Arrow's goodwill concept, possibly with different interpretations.

Let $[0, T]$ be the programming interval (with $T \in (0, +\infty]$), and let $\alpha > 0$ be the known value of the goodwill at the initial time. We assume that a firm aims at controlling the goodwill evolution in order to maximize its profit (discounted at rate $\rho \geq 0$)

$$J\left[u(t)\right] \;=\; \int_0^T \left[R\left(x(t)\right) - C\left(u(t)\right)\right] e^{-\rho t} dt + S\left(x(T)\right) e^{-\rho T}\,. \tag{4}$$

The function R, increasing and concave, represents the firm profit intensity, gross of the advertising costs. The function C, increasing and convex, represents the firm advertising costs intensity. Finally, the function S summarizes the effects of the final goodwill $x(T)$ on the profit to be obtained at time T or later on. It is consistent to assume that the scrap value vanishes, $S\left(x(T)\right) e^{-\rho T} = 0$, in problems with $T = +\infty$.

By choosing different specifications of the time horizon T and of the functions R, C and S, we can deal with a variety of advertising problems. We will present and solve some of them, in order to provide a general view, although certainly biased by our preferences and knowledge.

The consideration of nonlinear and convex advertising costs is sometimes translated into the assumption that the goodwill productivity term is nonlinear and concave in the advertising intensity [26], instead of being linear as in the original Nerlove-Arrow's equation (2). For instance, we may find a goodwill motion equation as (see e.g. [8], [40])

$$\dot{x}(t) \;=\; b\sqrt{u(t)} - \delta x(t)\,, \tag{5}$$

where b is a positive parameter, or as (see e.g. [51], [31])

$$\dot{x}(t) \;=\; b \ln u(t) - \delta x(t)\,, \quad (u(t) \geq 1)\,. \tag{6}$$

2.3 Discrete time models

The discrete time analogue of Nerlove-Arrow's dynamics is represented by the linear difference equation

$$x_t \;=\; u_t + (1-\delta)x_{t-1}\,. \tag{7}$$

It has a special practical importance, as empirical data are usually obtained as discrete time observations. It has been used recently, for instance, in a duopoly model [44] to investigate an automobile industry case. Little [42], concerned with the consistence of advertising models with empirical data, presented in 1975 *Brandaid*, a discrete time model which is a generalization of (7). In the same line, Zielske and Henry [60] in 1980 proposed the equation

$$x_t \;=\; \beta u_t + (1-\delta)x_{t-1}\,, \tag{8}$$

where $\beta > 0$, to account for the effect of TV advertising on the *awareness* of a product. Zielske and Henry call β the *memorization rate* and measure the advertising intensity u_t in *gross rating points* (GRP), a widely used indicator of population exposure to an advertising message [41, p.309].

3 Optimal advertising policies

In this Section we consider some typical optimal control problems, where an advertising intensity is looked for to optimize a suitable firm objective. Here a natural distinction arises between infinite and finite horizon problems. In the finite horizon class, a further distinction concerns the effect of goodwill on the objective functional: in some cases only the final goodwill value matters, whereas in other cases the goodwill path at all times is relevant. We will mention also some stochastic versions of the same optimal control problems.

3.1 Infinite horizon

Let $T = +\infty$, $R(x) = (q - \varepsilon_1)x^\gamma/\gamma - \varepsilon_2 x^{2\gamma}/2\gamma^2$, and $C(u) = \kappa u^2/2$, where $\gamma \in [1/2, 1)$, $q, \varepsilon_1, \varepsilon_2, \kappa > 0$ and $\varepsilon_1 < q$. We think of q as the sale price and x^γ/γ as the demand rate, so that qx^γ/γ is the revenue intensity. Furthermore, $\varepsilon_1 y + \varepsilon_2 y^2/2$ is the production cost intensity associated with the production intensity y. We obtain an instance of the classical Nerlove-Arrow's model which is particularly easy to study. The Pontryagin Maximum Principle necessary conditions [50, p. 234] lead us to study the saddle path of a system of ODEs.

The Hamiltonian function is

$$H(x, u, p, t) \; = \; (q - \varepsilon_1)x^\gamma/\gamma - \varepsilon_2 x^{2\gamma}/2\gamma^2 - \kappa u^2/2 + p(u - \delta x) \; , \qquad (9)$$

so that $H_{uu} < 0$ and an optimal control is unique and must satisfy the condition

$$\kappa u(t) \; = \; p(t) \, , \qquad (10)$$

provided that it exists. The optimality condition (10) represents the classical relation: *marginal advertising cost equals marginal goodwill value, i.e. marginal profit of goodwill.* It is worth recalling that p is the maximum price the firm is willing to pay to increase the goodwill by one unit. After substituting this information into the motion and adjoint equations we obtain the differential system

$$\begin{cases} \dot{x}(t) \; = \; p(t)/\kappa - \delta x(t) \, , \\ \dot{p}(t) \; = \; -(q - \varepsilon_1)x^{\gamma-1}(t) + \varepsilon_2 x^{2\gamma-1}(t)/\gamma + (\delta + \rho)\, p(t) \, . \end{cases} \qquad (11)$$

There exists a unique equilibrium point with coordinates (x^*, p^*) such that

$$p^* \; = \; \kappa \delta x^* \qquad (12)$$

and x^* is the solution of the equation

$$\gamma(q - \varepsilon_1)x^{\gamma-1} \; = \; \varepsilon_2 x^{2\gamma-1} + \gamma(\delta + \rho)\, \kappa \delta x \, . \qquad (13)$$

At the equilibrium, the optimal advertising policy $u^* = p^*/\kappa = \delta x^*$ is chosen precisely to compensate the goodwill decay.

3.2 Finite horizon - Advertising an event

Let $T < +\infty$, $R(x) = 0$, $C(u) = \kappa u^2/2$ and $S(x) = -\chi (x - \bar{x})^2/2$ (where κ, $\chi > 0$ and $\bar{x} > \alpha$). We obtain an instance of a model which differs from the classical Nerlove-Arrow's one because it does not account for sales before the final time T. The aim of this model is to program the advertising campaign for an event (or the launch of a product). Let us further assume that the discount rate is $\rho = 0$, as the advertising interval before the event takes place is short. In the particular case we present here we assume quadratic advertising costs $C(u)$ and a quadratic final penalty $S(x)$. The function $S(x)$ describes the payoff obtained by the organizers of an event like a concert or a theatre performance. For such events the number of available seats is a crucial parameter. We may think of a goodwill threshold $\bar{x}$ such that

– if the final goodwill exceeds it, $x(T) > \bar{x}$, then the demand is greater than the available seats, and there are some unsatisfied consumers;
– if the final goodwill is less than it, $x(T) < \bar{x}$, then the demand is less than the available seats, and some tickets remain unsold.

In the first case the organization suffers a loss of reputation, whereas in the latter a loss of revenue. The quadratic penalty function $S(x)$ is a symmetric representation of both these kinds of loss: it is a compromise between the analytical tractability of the problem and a precise description of the economic consequences of observing a demand level above or below the available seats. This problem has been studied in [7], [8], [27], [28], [43], in the context of the new product introduction. Moreover, a different model (with nonlinear deterministic dynamics) for the same problem has been recently analyzed in [32].

Under the previous assumptions, we have a linear quadratic deterministic optimal control problem and we can study it using the standard method of the completions of the squares [59]. After defining $z(t) = x(t) - \bar{x}$ the problem becomes

$$\begin{aligned} \min \int_0^T \kappa u(t)^2/2\,dt + \chi z(T)^2/2\,, \\ \dot{z}(t) \;=\; u(t) - \delta z(t) - \delta\bar{x}\,. \end{aligned} \tag{14}$$

We introduce the associated Riccati equations

$$\begin{cases} \dot{q}(t) \;=\; 2\delta q(t) + q^2(t)/\kappa\,, \\ \dot{s}(t) \;=\; (\delta + q(t)/\kappa)\,s(t) + \delta\bar{x}q(t)\,, \\ q(T) = \;\chi\,, \\ s(T) = \;0\,. \end{cases} \tag{15}$$

If there exists a solution of the ODEs system (15), then there exists a unique feedback optimal control that can be written as

$$u^*(t) \;=\; -\left\{q(t)\left(x^*(t) - \bar{x}\right) + s(t)\right\}/\kappa\,. \tag{16}$$

The interpretation of the result is immediate in the special case that $\delta = 0$, which is a reasonable assumption as far as the advertising campaign is short. In this case the ODEs system (15) has the solution $s(t) \equiv 0$, $q(t) = \chi\kappa/\left(\kappa + \chi\left(T - t\right)\right)$, and the optimal control is $u^*(t) = -\chi\left(x^*(t) - \bar{x}\right)/\left(\kappa + \chi\left(T - t\right)\right)$. The function $q(t)$ is strictly increasing: the weight of the reaction to the deviation of the goodwill from the target, $x^*(t) - \bar{x}$, is higher and higher as time approaches T.

This same problem has been studied in [4], where the goodwill is assumed to satisfy a linear stochastic differential equation. There the control (advertising intensity) enters directly the diffusion term of the motion SDE. Therefore, some uncertainty is introduced in the advertising effectiveness and this fact modifies the structure of the optimal solution. It is interesting to compare the optimal use of a deterministic advertising channel (as the one analyzed in this example) with the one of an advertising channel which has also a stochastic effect on the goodwill evolution.

3.3 Finite horizon - Sale of a seasonal product

An interesting variant of the finite horizon problem concerns the sale of a seasonal product. Here we introduce a second state variable $y(t)$ to represent the product sales and assume that the demand is a linear function of the goodwill

$$\dot{y}(t) \;=\; x(t)\,, \quad y(0) \;=\; 0\,. \tag{17}$$

Let $C(u) = \kappa u^2/2$, $\kappa > 0$, as for the above problem, and let us consider explicitly the constraint $u \geq 0$ on the advertising intensity. Let $R(x) = qx$ and $S(x, y) = -w(y)$, where $q > 0$ is the price of the product and $w(y)$ is a positive, increasing and convex function, representing the production cost of the quantity y of product. Let us further assume that the discount rate is $\rho = 0$, as the sale interval is short for a seasonal product.

For the present problem we can use the Pontryagin Maximum Principle [50, p. 85 and p. 182] in order to characterize an open loop optimal control. The Hamiltonian function is

$$H(x, y, u, p_1, p_2, t) \;=\; qx - \kappa u^2/2 + p_1(u - \delta x) + p_2 x\,, \tag{18}$$

so that $H_{uu} < 0$ and the unique optimal control must be

$$u(t) \;=\; \max\{0,\, p_1(t)/\kappa\}\,, \tag{19}$$

provided that it exists. As in the infinite horizon case we find that, at the optimum, marginal advertising cost equals marginal goodwill value, $\kappa u(t) = p_1(t)$, provided that advertising is effective, $u(t) > 0$. From the adjoint equations and the trasversality conditions, we obtain that

$$p_1(t) = \frac{q + p_2}{\delta}\left(1 - e^{\delta(t-T)}\right)\,, \tag{20}$$

$$p_2(t) = p_2 = -w'(y(T)) < 0\,. \tag{21}$$

Therefore, after defining

$$y_0 \;=\; \frac{\alpha}{\delta}\left(1 - e^{-\delta T}\right)\,, \tag{22}$$

as the *free* sale level, which is attainable without any advertising effort, we have that

- either $q \leq w'(y_0)$, and then $u^*(t) \equiv 0$, $y^*(T) \equiv y_0$;
- or there exists $y^* > y_0$ such that $-p_2 = w'(y^*) < q$, and then $u^*(t) > 0$ for all $t < T$, $y^*(T) = y^*$ ($u^*(t)$ is strictly decreasing and $u^*(T) = 0$).

Problems of this kind, but with linear advertising costs and, on the other hand, with further constraints on the advertising intensity (control) $u(t)$, have been studied in [16], [17], [18], [19], [20], [9], [21]: because of the special problem structure the optimal policies are bang-bang controls. In this research stream are also the papers [10], [11], where word-of-mouth effects, saturation aversion and different behaviors of consumers and retailers are taken into account.

4 Several agents in the distribution channel

The models considered in the previous sections assumed that the demand depended on the goodwill only, i.e. essentially on the firm marketing strategies. In practice, the demand depends also on the competitors actions. Hence a good model should account for the competition among the different agents acting in the market. One firm profit problem is part of a dynamic game, where firms are the players, each one with its own objective and available strategies.

Each player wants to maximize his payoff (total discounted profit/utility), which depends on his own strategy as well as on the strategies of all other players. Let N be the number of firms and let $u_i(t)$, $i \in \{1, \ldots, N\}$, be the advertising intensity of the i-th firm (i.e. the i-th firm strategy). Typically, two kinds of game formulations can be based on the linear model for the goodwill evolution.

I) There is one goodwill variable for each player/firm and it is affected by the advertising strategy of the same firm.
Let $x_i(t)$, $i \in \{1, \ldots, N\}$, be the goodwill stock of the i-th firm, with dynamics described by the differential equation

$$\begin{cases} \dot{x}_i(t) = & u_i(t) - \delta x_i(t) \ , \\ x_i(0) = & \alpha_i \, . \end{cases} \tag{23}$$

The payoff of the $i-th$ player has the form

$$J_i\left[u(t)\right] \ = \ \int_0^T \left[R_i\left(x(t)\right) - C_i\left(u(t)\right)\right] e^{-\rho t} dt + S_i\left(x(T)\right) e^{-\rho T} \ , \tag{24}$$

where $x(t)$ and $u(t)$ are the n-dimensional goodwill, and advertising strategy variables.

II) There is a unique goodwill variable, which is affected by the advertising efforts (controls) of the different firms

$$\begin{cases} \dot{x}(t) = & \sum_{i=1}^{N} u_i(t) - \delta x(t) \ , \\ x(0) = & \alpha \, . \end{cases} \tag{25}$$

The payoff of the $i-th$ player has still the form (24), where $x(t)$ is now a 1-dimensional goodwill variable.

The optimal control problems arising from the two formulations can be solved using either the Pontryagin Maximum Principle or the Hamilton Jacobi Bellman Equation approach. The first one leads to open-loop strategies/controls which only depend on time, while the second one leads to closed-loop strategies which depend also on the state variables. Unlike the one-player optimal control problems where the two approaches lead to the same optimal solutions, in a differential game different representations of the feasible control paths lead to different optimization problems for the players.

The use of the Hamilton Jacobi Bellman equation approach leads one to find the so-called Markov Nash equilibrium of a differential game, i.e. the strategy which is also an equilibrium of any subgame that starts out on the equilibrium trajectory.

The "advertising goodwill models" [37] have been tackled using either formulation. As far as the type I formulation is concerned, the first examples of use of the Nerlove-Arrow's model are given by [56] and [25], within different oligopoly contexts. In the former the demand of each one of two firms depends on the goodwill of both firms. In the latter the market share of each one of N firms is assumed to be a function of the goodwill of all firms. Other examples of a type I game are given by [13] and [12]: in the latter a model of advertising competition is discussed in a dynamic duopoly with diminishing returns to advertising effort. More recently we find [45] and [46].

Examples of type II games are given by [34], [33], [35] and [36], which consider a distribution channel, where a retailer promotes the manufacturer product and the latter may possibly spend in advertising to sustain the retailer campaign. These are instances of the so-called *leader-follower* model: a two-player game where one of the players (the manufacturer - leader) makes his decision before the other player (the retailer - follower) makes hers. The solution of such games is sought in the form of a Stackelberg equilibrium [15, p.111].

We have a *linear state differential game* [15, p.187] when both the system dynamics and the utility functions are linear in the state variables and there is no multiplicative interaction between the state and the control variables. The control paths of a linear state game are uniquely determined by the costate trajectories and the adjoint equations, together with the transversality condition, do not involve the state variables. In particular the open-loop strategies are independent of the initial state value and the open-loop Nash equilibria are Markov equilibria too. These same qualitative characteristics are possessed by some other games. An example is a game with Nerlove-Arrow's dynamics which is provided by [15, p.191]. Here the two players' goodwill motion equations are

$$\dot{x_i}\,(t) \;=\; u_i\,(t) - \delta_i(u_j(t))\; x_i\,(t)\;, \quad i,j \in \{1,2\}\,,\; i \neq j\,, \tag{26}$$

where $\delta_i(u_j)$, the decay parameter of the i-th firm goodwill, depends on the other firm advertising strategy. The i-th firm payoff, $i \in \{1,2\}$, is

$$J_i\,[u(t)] \;=\; \int_0^T \left[\pi_i x_i(t) - C_i(u_i(t))\right] e^{-\rho t} dt + \sigma_i x_i(T) e^{-\rho T}\,, \tag{27}$$

where $\pi_i > 0$ measures the profitability of the i-th goodwill stock and $\sigma_i > 0$ is the marginal revenue of the final goodwill. The game possesses the linear state games property that its open-loop Nash equilibria are Markov perfect. In fact, after denoting by $V_i(x_1, x_2, t)$ the i-th firm value function, the Markov Nash equilibria of the game are characterized by the Hamilton-Jacobi-Bellman equations

$$\rho V_i(x_1,x_2,t) - \frac{\partial V_i(x_1,x_2,t)}{\partial t} =$$

$$= \max\left\{\pi_i x_i - C_i(u_i) + \frac{\partial V_i(x_1,x_2,t)}{\partial x_i}\left[u_i - \delta_i(u_j(t))x_i\right]\right. \tag{28}$$

$$\left. + \frac{\partial V_i(x_1,x_2,t)}{\partial x_j}\left[u_j - \delta_j(u_i(t))x_j\right] \mid u_i \in \Re\right\},$$

with terminal conditions

$$V_i(x_1,x_2,T) = \sigma_i x_i(T), \quad i \in \{1,2\}. \tag{29}$$

Eventually a solution $(u_1(t),u_2(t))$ is characterized, which depends on time t, but is independent of the state $(x_1(t),x_2(t))$, i.e. the Markov Nash equilibrium just obtained is an open-loop solution.

Early applications of linear state differential games to advertising can be found in [31] and [22].

5 Market segmentation

Quite often a consumer population cannot be considered homogeneous by a firm which wants to advertise and sell a product. In fact, the firm has to determine the suitable *target market* [39, p.379], i.e. the part of the consumers which may have an interest in buying the product. To this goal, the firm divides the market into distinct *segments*, consumer groups which exhibit special needs and behaviors [39, p.379] and which require specific products and marketing mix. Then the firm has to decide to which consumer groups the product should be proposed and how to reach each segment using the available marketing tools, while considering that different advertising channels entail specific costs and different segments offer different marginal revenues.

So far, only few papers on optimal control applications to advertising have focused on segmented markets (see e.g. [49], [57], [28]), as they exhibit a higher complexity than the homogeneous ones. Of course, the simplicity of Nerlove-Arrow's model makes it a first candidate to that goal.

Let the consumer population Ω be partitioned into segments Ω_a, $a \in A$, where A is the finite set of segment labels. Let $x(t,a)$ represent the stock of goodwill of the product at time $t \in [0,T]$, for the (consumers in the) a segment. Here we assume that the goodwill evolution satisfies the set of independent ordinary differential equations

$$\dot{x}(t,a) = u(t,a) - \delta(a)\,x(t,a), \qquad a \in A, \tag{30}$$

where $\delta(a) > 0$ represents the goodwill depreciation rate for the members of the consumer group a and $u(t,a)$ is the effective advertising intensity at time t

directed to that same group. Moreover, we assume we know the goodwill level at the initial time for all segments,

$$x\,(0,a) \;=\; \alpha\,(a) \;\geq\; 0\,. \tag{31}$$

Under such assumptions a new product introduction problem has been studied in [5], with objective functional

$$J \;=\; -\sum_{a\in A}\int_0^T c(a,u(t,a))\,dt + \sum_{a\in A}\pi\,(a)\,x\,(T,a)\;, \tag{32}$$

whose first term represents advertising costs, whereas the second one represents revenue (it replaces $S(x(T))$ in (4), $\pi\,(a)\geq 0$ is the marginal revenue of goodwill for segment a). The further question of selecting a special advertising channel for a segmented market has been considered in [6]. This is a practical issue, whose importance in real life situations is particularly evident when marketing budgets cuts are imposed by general economic conditions (e.g. fears of economic slowdown [30]). The generalized linear (Nerlove-Arrow's) framework allows to provide an answer to it in terms of a *channel preference index*.

A special kind of segmentation is considered in [11], where the goodwill has a component for the consumers and one for the retailers.

A further step in the analysis of different behaviors of a population is addressed by the *age-segmentation*, which has been profitably considered in contexts different from that of advertising, e.g. social analysis and drug addiction (see [1], [29]). In this case, we are led to consider an age-distributed goodwill variable which evolves according to a linear partial differential equation

$$\begin{array}{l} \partial_t y\,(t,a) + \partial_a y\,(t,a) \;=\; u\,(t,a) - \delta\,(a)\,y\,(t,a)\;, \\ y\,(0,a) \;=\; \alpha\,(a)\;, \\ y\,(t,0) \;=\; \beta\,(t)\;, \end{array} \tag{33}$$

where $\delta\,(a)\geq 0$ represents the depreciation rate for the members of age a and $u\,(t,a)\geq 0$ is the effective advertising intensity at time t directed to the members of age a, $\alpha\,(a)\geq 0$ is the known goodwill level at the initial time for the age a class of the population and $\beta\,(t)$ is the goodwill level at all times for the age 0 class. This is the natural extension of the Nerlove-Arrow's model.

The optimal control tools developed recently for the treatment of those application models [24], have made it possible to study an extension of the new product introduction problem [28], where an age-sensitive product is considered. Such a problem has the simplest kind of objective functional. In the same direction, other and more complex advertising problems may be addressed, using the Feichtinger-Tragler-Veliov's optimality conditions, as far as the generalized Nerlove-Arrow's model is adopted.

6 Several communication tools

The simultaneous use of different communication tools, known as *integrated marketing communication*, amplifies the effects that each one could have if used alone (see [48], [54]). Furthermore, over the last years the market has deeply changed; we observe the multiplication of the TV channels, technical magazines and journals. This fact has contributed to the increased need of integration among the communication tools, such as sales promotion, public relations, sponsorship, ... [38]. Therefore it becomes important to adopt an integrated strategy with different communication tools.

In order to account for the effects of different communication tools, in particular advertising channels, we may introduce one control function (channel activation intensity) $u_i(t)$ for each tool and substitute the advertising investment term $u(t)$ in (2) with a function of all controls $f(u_1(t), u_2(t), \ldots, u_n(t))$. The simple assumption that $f(u_1, u_2, \ldots, u_n)$ is linear, leads to the equation

$$\dot{x}(t) = \sum_{i=1}^{n} \beta_i u_i(t) - \delta x(t) , \tag{34}$$

where β_i are positive parameters. In the framework of game theory, the equation (34) has been used to model the manufacturer-retailer interaction (see [34], [36]). In the framework of optimal control, it has been used to model the use of different communication instruments by a single firm (see [2], [3]).

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